DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
MATH 220, EXAM 2, FALL 2010

Your Name: ______________________________

On this exam, you may use a calculator and a formula page. It is not sufficient to just write down the answers. You must explain how you arrived at your answers and how you know they are correct.

1	(40)	________
2	(40)	________
3	(40)	________
4	(40)	________
5	(40)	________
bonus	(30)	________
Total	(200)	________

- **(40 points)** A 16-pound weight stretches a spring 2 feet. The medium through which the weight moves offers a resistance numerically equal to 4 times the velocity. Find the motion if the weight is released from a point one foot below the equilibrium position with zero velocity.

First we will determine the constants m, γ, k from the given data.

$m = \frac{16}{32} = \frac{1}{2}$; $\gamma = 4$; $k = \frac{16}{2} = 8$

10 points

The equation is $\begin{cases} \frac{1}{2}u'' + 4u' + 8u = 0 \\ u(0) = 1 \\ u'(0) = 0 \end{cases}$

Solve: $r^2 + 8r + 16 = 0$

$r_{1,2} = -4$ double root, critically damped

$$\boxed{u(t) = c_1 \cdot e^{-4t} + c_2 \cdot t \cdot e^{-4t}}$$

20 points

Find the constants based on the initial conditions.

$u(0) = c_1 = 1 \Rightarrow u(t) = e^{-4t} + c_2 \cdot t \cdot e^{-4t}$

$u'(t) = -4e^{-4t} + c_2 \cdot e^{-4t} - 4c_2 te^{-4t}$

$u'(0) = -4 + c_2 = 0 \Rightarrow c_2 = 4$

$$\boxed{u(t) = e^{-4t} + 4te^{-4t}}$$

10 points

- (~~30~~ 40 points) Find the solution of the initial value problem using the Laplace transform.

$$\begin{cases} y'' - 4y' + 4y = 0 \\ y(0) = 1, y'(0) = 1 \end{cases}$$

Apply Laplace transform on both sides, $\mathcal{L}\{y(t)\} = Y(s)$:

$$s^2 Y(s) - s y(0) - y'(0) - 4(s\cdot Y(s) - y(0)) + 4Y(s) = 0$$

$$s^2 Y(s) - s - 1 - 4s Y(s) + 4 + 4Y(s) = 0$$

$$\boxed{(s^2 - 4s + 4)Y(s) = s - 3}$$ $\boxed{\text{20 points}}$

$$Y(s) = \frac{s-3}{s^2 - 4s + 4} = \frac{s-3}{(s-2)^2} = \frac{s-2}{(s-2)^2} - \frac{1}{(s-2)^2}$$

$$\boxed{Y(s) = \frac{1}{s-2} - \frac{1}{(s-2)^2}}$$ $\boxed{\text{10 points}}$

Find the inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y(t) = e^{2t} - t\cdot e^{2t}}$$ $\boxed{\text{10 points}}$

40

- (~~2~~0 **points)** Find the inverse Laplace transform of the function

$$F(s) = \frac{se^{-3s}}{(s+1)(s^2+4)}.$$

$$\mathcal{L}^{-1}\left\{e^{-3s} \cdot \frac{s}{(s+1)(s^2+4)}\right\} = u_3(t) \cdot f(t-3),$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\}$

$\boxed{10 \text{ points}}$

Now to find $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\}$ split into simple fractions.

$$\frac{s}{(s+1)(s^2+4)} = \frac{a}{s+1} + \frac{bs+c}{s^2+4}$$

$\boxed{10 \text{ points}}$

$$a(s^2+4) + (bs+c)(s+1) = s$$

s^2: $a + b = 0$
s: $b + c = 1$
s^0: $4a + c = 0$

solve for a, b, c to get

$4a - b = -1$
$a + b = 0$
$5a = 1$

$\boxed{a = \frac{-1}{5},\ b = +\frac{1}{5}}$

$\boxed{10 \text{ points}} \rightarrow \boxed{c = \frac{4}{5}}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{5} \cdot \frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{\frac{1}{5}s + \frac{4}{5}}{s^2+4}\right\} =$$

$$= -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4}\right\}$$

$\boxed{f(t) = -\frac{1}{5}e^{-t} + \frac{1}{5}\cos 2t + \frac{1}{5}\sin 2t}$ $\boxed{10 \text{ points}}$

- **(40 points)** Find the solution of the initial value problem and describe its behavior for increasing t. Use Laplace transform.

$$\begin{cases} y'' - 2y' + 2y = e^{-t} \\ y(0) = 0, y'(0) = 1 \end{cases}$$

Apply Laplace transform on both sides to get

$$s^2 Y(s) - s \cdot y(0) - y'(0) - 2(s \cdot Y(s) - y(0)) + 2Y(s) = \frac{1}{s+1}$$

$$s^2 \cdot Y(s) - 1 - 2sY(s) + 2Y(s) = \frac{1}{s+1}$$

$$\boxed{(s^2 - 2s + 2)\, Y(s) = 1 + \frac{1}{s+1}} \qquad \boxed{10 \text{ points}}$$

$$Y(s) = \frac{s+2}{s+1} \cdot \frac{1}{s^2 - 2s + 2} = \frac{s+2}{(s+1)(s^2 - 2s + 2)}$$

represent as simple fractions:

$$\frac{s+2}{(s+1)(s^2-2s+2)} = \frac{a}{s+1} + \frac{bs + c}{s^2 - 2s + 2} \qquad \boxed{20 \text{ points}}$$

$$a(s^2 - 2s + 2) + (bs + c)(s+1) = s + 2$$

s^2: $a + b = 0$

s: $-2a + b + c = 1$ $\}$ $-$

s^0: $2a + c = 2$

$a + b = 0$

$4a - b = 1$

$5a = 1 \Rightarrow \boxed{a = \frac{1}{5}}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{5} \cdot \frac{1}{s+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s-8}{s^2 - 2s + 2}\right\} \qquad \boxed{c = 2 - 2a = \frac{8}{5}} \qquad \boxed{b = -\frac{1}{5}}$$

$\downarrow \frac{1}{5}e^{-t}$

$$\frac{s-8}{s^2-2s+2} = \frac{s-1}{(s-1)^2+1} - 7 \cdot \frac{1}{(s-1)^2+1} \xrightarrow{\mathcal{L}^{-1}} e^t \cos t - 7e^t \sin t$$

$$\boxed{y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^t \cos t + \frac{7}{5}e^t \sin t} \qquad \boxed{10 \text{ points}}$$

- **(40 points)** Find the eigenvalues and the eigenvectors of the matrix

$$\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}.$$

To find to eigenvalues solve

$$\begin{vmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)(4-\lambda)+5 = \lambda^2-2\lambda-3 = 0$$

$\boxed{\text{10 points}}$

$$\boxed{\begin{matrix}\lambda_1 = 3 \\ \lambda_2 = -1\end{matrix}}$$ eigenvalues

$\boxed{\text{10 points}}$

For the first eigenvector $x^{(1)} = \begin{pmatrix} u \\ v \end{pmatrix}$ we solve

$$(A-3I)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2-3 & 1 \\ -5 & 4-3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-5u+v=0$$

$$\Rightarrow \boxed{x^{(1)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}}$$

$\boxed{\text{10 points}}$

For the second eigenvector $x^{(2)} = \begin{pmatrix} u \\ v \end{pmatrix}$

$$(A+I)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2+1 & 1 \\ -5 & 4+1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-u+v=0$$

$$\Rightarrow \boxed{x^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\boxed{\text{10 points}}$

- **(30 points) bonus problem** Use Laplace transform to find the solution of the initial-value problem.

$$\left| \begin{array}{l} y^{(4)} + 5y'' + 4y = 1 - u_\pi(t) \\ y(0) = 0, y'(0) = 0, y"(0) = 0, y'''(0) = 0 \end{array} \right.$$