DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
MATH 220 - FALL 2010 - FINAL EXAM

Your Name:

On this exam, you may use a calculator and some formula notes.

It is not sufficient to just write down the answers. You must explain how you arrived at your answers and how you know they are correct.

1	(30)	30
2	(30)	30
3	(30)	30
4	(30)	30
5	(30)	30
6	(30)	30
7	(30)	30
8	(30)	30
9	(30)	30
10	(30)	30
Total	(300)	300

Great work!

$\boxed{A}$

- **1. (30 points)** Solve the initial-value problem and draw the graph of the solution int. factor

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \quad y(\pi) = 0.$$

$\boxed{y = \frac{\sin t}{t^2}}$

$\mu = e^{2\int \frac{1}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$

$\therefore y't^2 + 2yt = \cos t$

$\therefore (t^2 y)' = \cos t$

$\therefore t^2 y = \int \cos t \, dt$

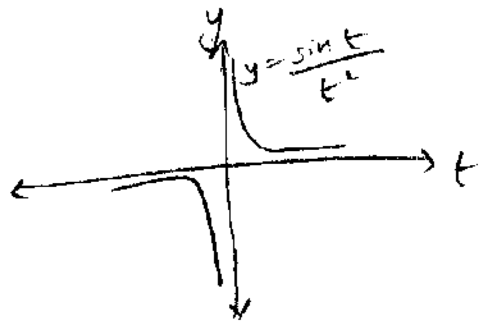
$t^2 y = \sin t + c$

$\boxed{y = \frac{\sin t}{t^2} + \frac{c}{t^2}}$ gen. soln.

$y(\pi) = \frac{\sin \pi}{\pi^2} + \frac{c}{\pi^2}$

$= \frac{c}{\pi^2} = 0$

$\therefore \boxed{c = 0}$



V

- **2. (30 points)** Solve the initial-value problem

sep

$$\begin{cases} x + ye^{-x}y' = 0 \\ y(0) = 1 \end{cases} \Rightarrow x + ye^{-x}\frac{dy}{dx} = 0$$

$\therefore x\,dx + y\,dy\,e^{-x} = 0$

$\therefore xe^{x}dx + y\,dy = 0$

$\therefore xe^{x}dx = -y\,dy$

$\int xe^{x}dx = -\int y\,dy$

let $u = x \Rightarrow du = dx$

$dv = e^{x}dx \Rightarrow v = e^{x}$

$uv - \int v\,du$

$= xe^{x} - \int e^{x}dx$

$= xe^{x} - e^{x}$

$\therefore xe^{x} - e^{x} = -\frac{y^2}{2} + c$

$\therefore 2xe^{x} - 2e^{x} = -y^2 + 2c$

$\therefore y^2 + 2xe^{x} - 2e^{x} = 2c$

$\therefore y^2 = 2e^{x} - 2xe^{x} + 2c$

$\therefore y = \pm\sqrt{2e^{x} - 2xe^{x} + 2c}$

$y(0) = \pm\sqrt{2 + 2c} = 1$

$\therefore c = -\frac{1}{2}$ & $\boxed{y(t) = \sqrt{2e^{x} - 2xe^{x} - 1}}$

V

$2\left(\frac{-1}{2}\right.$

$2 + 2c = 1$

$2c = -1$

$c = -\frac{1}{2}$

$2 + 2\left(-\frac{1}{2}\right)$

$+\ 2 - 1 = 1$

$x^2 \le \frac{28}{3}$ 9.3

$a = 1$
$b = -x$
$c = x^2 - 7$

$28 - 3x^2 \ge 0$
$-3x^2 \ge -28$
$3x^2 \le 28$

• **3. (30 points)** Find the explicit solution of the initial-value problem and determine approximately where the solution is valid

$$(2x - y)dx + (2y - x)dy = 0, \quad y(1) = 3$$

$\underbrace{(2x-y)}_{M} + \underbrace{(2y-x)}_{N}y' = 0$

$M_y = -1$, $N_x = -1$ $\Rightarrow$ exact ✓

$\therefore \psi(x,y) = c$ s.t. $\psi_x = 2x - y$ & $\psi_y = 2y - x$

$\psi_x = 2x - y$

$\therefore \psi = \int (2x - y)dx$

$= \frac{2x^2}{2} - xy + c(y)$

$\psi = x^2 - xy + c'(y)$

$\psi_y = \frac{\partial}{\partial y}(x^2 - xy + c(y))$

$\psi_y = -x + c'(y) = -x + 2y$

$\therefore c'(y) = 2y$

$\therefore c(y) = \int 2y\,dy = \frac{2y^2}{2} + k = y^2 + k$

$\psi = x^2 - xy + y^2 + k = c$

$x^2 - xy + y^2 = c$

$x = 1,\ y = 3$

$1 - 3 + 9 = c \Rightarrow \boxed{c = 7}$

$\boxed{\psi = x^2 - xy + y^2 = 7}$

$y^2 + x^2 - xy - 7 = 0$

$y^2 - xy + \underbrace{x^2 - 7}_{} = 0$

$y = \frac{x \pm \sqrt{x^2 - 4(x^2 - 7)}}{2}$

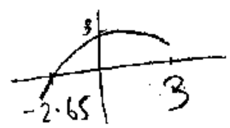
$y = \frac{x \pm \sqrt{x^2 - 4x^2 + 28}}{2}$

$\boxed{y = \frac{x \pm \sqrt{28 - 3x^2}}{2}}$

$y(1) = 3$

$\therefore 3 = \frac{1 \pm \sqrt{28 - 3}}{2} \Rightarrow \frac{1 + \sqrt{28-3}}{2}$

$\boxed{y = \frac{x + \sqrt{28 - 3x^2}}{2}}$



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the sol is valid approx between $[-3, 3]$

• **4. (30 points)** Solve the initial-value problem, sketch the graph of the solution and describe its behavior for increasing t.

$$y'' - 2y' + 5y = 0$$
$$y(\pi/2) = 0,\ y'(\pi/2) = 2$$

$(\frac{-1}{c}, 0)$

$r^2 - 2r + 5 = 0$

$r = \frac{2 \pm \sqrt{4 - 20}}{2}$

$= \frac{2 \pm 4i}{2}$

$= 1 \pm 2i \Rightarrow \lambda = 1, \mu = 2$

$\boxed{y = c_1 e^t \cos(2t) + c_2 e^t \sin 2t}$

$y(\frac{\pi}{2}) = c_1 e^{\pi/2}\cos(\pi) + c_2 e^{\pi/2}\sin(\pi)$, $\sin(\pi) = 0$

$= -c_1 e^{\pi/2} = 0$

$\therefore \boxed{c_1 = 0} \Rightarrow y = c_2 e^t \sin 2t$

$y'(t) = c_2 e^t \sin 2t + 2c_2 e^t \cos 2t$

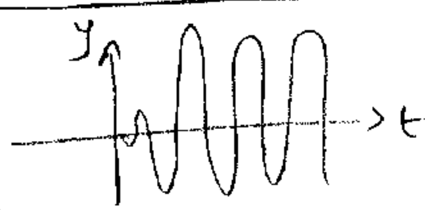
$y'(\frac{\pi}{2}) = c_2 e^{\pi/2}\sin(\pi) + 2c_2 e^{\pi/2}\cos(\pi)$, $\sin(\pi) = 0$

$= -2c_2 e^{\pi/2} = 2$

$\therefore -2c_2 = \frac{2}{e^{\pi/2}} \Rightarrow c_2 = -\frac{1}{e^{\pi/2}}$

$c_2 = -1/e^{\pi/2}$

$\boxed{y(t) = \frac{-1}{e^{\pi/2}} e^t \sin 2t} = -e^{(t - \pi/2)}\sin 2t$



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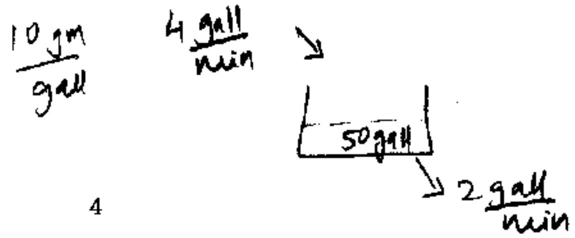
for increasing t, the solution has an oscillating graph. The amplitude is small initially & it increases as t increases



1 min : gain 2 gall
t min : 2t

- **5. (30 points)** A tank with a capacity of 100 gallons is half full of fresh water. A pipe is opened that lets treated sewage enter the tank at the rate of 4 gallons per minute. At the same time, a drain is opened to allow the mixture to leave the tank at the rate of 2 gallons per minute. If the treated sewage contains 10 grams of unstable potassium per gallon, what is the concentration of potassium in the tank when it is full?

Let Q = amt of pot. in tank at time t (g) . $Q(0) = 0\,g$.

$\frac{dQ}{dt}$ = rate$_{in}$ − rate$_{out}$

$= \left(\frac{10\,g}{\text{gall}} \times 4\frac{\text{gall}}{\text{min}}\right) - \left(\frac{Q\ g}{(50+2t)\,\text{gall}} \times \frac{2\,\text{gall}}{\text{min}}\right) \Rightarrow Q' = 40 - \frac{2Q}{50+2t}$

$\therefore Q' + \frac{2Q}{50+2t} = 40$

$\mu = e^{2\int \frac{1}{50+2t}dt}$

$2\int \frac{1}{50+2t}dt$

let $u = 50+2t$

$\frac{du}{dt} = 2 \Rightarrow dt = \frac{du}{2}$

$\therefore 2\int \frac{1}{u}\cdot\frac{du}{2} = \ln u$

$= \ln|2t+50|$

$\therefore \mu = e^{\ln|2t+50|} = 2t+50$

$\therefore Q'(2t+50) + \frac{2Q}{\cancel{50+2t}}\cancel{(50+2t)} = 40(50+2t)$

$\therefore Q'(2t+50) + 2Q = 2000 + 80t$

$\therefore \left((2t+50)Q\right)' = 80t + 2000$

$(2t+50)Q = \int 80t + 2000\,dt$

$= \frac{\overset{40}{\cancel{80}}t^2}{\cancel{2}} + 2000t + C$

$= 40t^2 + 2000t + C$

$\therefore Q = \frac{40t^2}{2t+50} + \frac{2000t}{2t+50} + \frac{C}{2t+50}$

$Q(0) = 0$

$\therefore 0 = \frac{C}{50} \Rightarrow C = 0$

$Q = \frac{40t^2}{2t+50} + \frac{2000t}{2t+50} \Rightarrow \boxed{Q = \frac{40t^2+2000t}{2t+50}}$

Tank full $\Rightarrow$

$50 + 2t = 100$

$2t = 50$

$t = 25$ min.

$Q(25) = \frac{40(25)^2 + 2000(25)}{2(25)+50}$

$= \frac{25000 + 50000}{100} = 750$

$\therefore$ conc of pot. when tank is full $= 750\,g$

✓

$12B - 9C = 0$
$-12B - 16C = 16$
$-25C = 16$

• **6. (30 points)** Find the general solution of the non-homogeneous equation

$$y'' + 2y' + y = 3 + 4\sin 2t$$

$y_p = A + B\sin 2t + C\cos 2t$
$y_p' = 2B\cos 2t - 2C\sin 2t$
$y_p'' = -4B\sin 2t - 4C\cos 2t$
$-4B - 4C + B = 4 \Rightarrow -3B - 4C = 4$
$-4C + 4B + C = 0 \Rightarrow 4B - 3C = 0$
$A = 3$

a) $y'' + 2y' + y = 0$
$\therefore r^2 + 2r + 1 = 0$
$r = \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2}{2} = -1$
$\therefore \boxed{y_h = c_1 e^{-t} + c_2 t e^{-t}}$

b) $y_{p_1} = A$
$\therefore y_{p_1}' = 0$
$y_{p_1}'' = 0$
$\therefore 0 + 0 + A = 3$
$\therefore A = 3$

$y_{p_2} = B\sin 2t + C\cos 2t$
$y_{p_2}' = 2B\cos 2t - 2C\sin 2t$
$y_{p_2}'' = -4B\sin 2t - 4C\cos 2t$
$\therefore -4B\sin 2t - 4C\cos 2t + 4B\cos 2t - 4C\sin 2t + B\sin 2t + C\cos 2t = 4\sin 2t$

$$\left.\begin{array}{l} -3B - 4C = 4 \\ 4B - 3C = 0 \end{array}\right\} \Rightarrow \begin{array}{l} B = -12/25 \\ C = -16/25 \end{array}$$

$\therefore y_{p_2} = \frac{-12}{25}\sin 2t - \frac{16}{25}\cos 2t$

$\therefore \boxed{y_p = 3 - \frac{12}{25}\sin 2t - \frac{16}{25}\cos 2t}$

$\therefore y(t) = y_h + y_p$

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t} + 3 - \frac{12}{25}\sin 2t - \frac{16}{25}\cos 2t}$$

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$v = \frac{3}{2}u$

$1 - -1$ $\qquad -4+2$

- **7. (30 points)** Find the general solution of the linear system, sketch the graph and describe the behavior as $t \to \infty$.

$$\begin{cases} x' = x - 2y \\ y' = 3x - 4y. \end{cases} \Rightarrow X' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} X$$

$\begin{vmatrix} (1-\lambda) & -2 \\ 3 & (-4-\lambda) \end{vmatrix} = (1-\lambda)(-4-\lambda) + 6 = -4 - \lambda + 4\lambda + \lambda^2 + 6 = \lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = -1, -2$

$\therefore$ for $\lambda = -1$

$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right] \Rightarrow \left.\begin{array}{l} 2u - 2v = 0 \\ \therefore 2u = 2v \\ \therefore u = v \end{array}\right\} \Rightarrow \boxed{\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

for $\lambda = -2$

$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right] \Rightarrow 3u = 2v \Rightarrow \boxed{\xi_2 = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}}$

$\therefore \boxed{X = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} e^{-2t}}$

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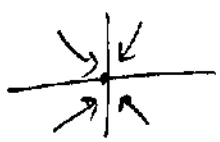

as $t \to \infty$, $X \to 0$

Sink

- **8. (30 points)** Find the solution of the initial-value problem

Laplace

$$y^{(4)} - y = 0; \quad y(0) = 1, \; y'(0) = 0, \; y''(0) = -2, \; y'''(0) = 0$$

$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0$

$s^4 Y(s) - s^3 + 2s - Y(s) = 0$

$Y(s)(s^4 - 1) = s^3 - 2s$

$Y(s) = \dfrac{s^3 - 2s}{s^4 - 1} = \dfrac{s^3 - 2s}{(s-1)(s+1)(s^2+1^2)} = \dfrac{A}{s-1} + \dfrac{B}{s+1} + \dfrac{Cs + D}{s^2 + 1^2} = \dfrac{A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+1)}{(s-1)(s+1)(s^2+1^2)}$

$= \dfrac{(As + A)(s^2+1) + (Bs - B)(s^2+1) + (Cs^2 - Cs + Ds - D)(s+1)}{(s-1)(s+1)(s^2+1^2)} = \dfrac{As^3 + As + As^2 + A + Bs^3 + Bs - Bs^2 - B + Cs^3 + \cancel{Cs^2} - \cancel{Cs^2} - Cs + Ds^2 + \cancel{Ds} - \cancel{Ds} - D}{(s-1)(s+1)(s^2+1^2)}$

$$\left.\begin{array}{ll} s^3 & A + B + C = 1 \\ s^2 & A - B + D = 0 \\ s & A + B - C = -2 \\ & A - B - D = 0 \end{array}\right\} \begin{array}{l} A = -1/4 \\ B = -1/4 \\ C = 3/2 \\ D = 0 \end{array} \Rightarrow Y(s) = \frac{-1}{4}\left(\frac{1}{s-1}\right) - \frac{1}{4}\left(\frac{1}{s+1}\right) + \frac{\frac{3}{2}s}{(s^2+1^2)}$$

$$= \frac{-1}{4}\left(\frac{1}{s-1}\right) - \frac{1}{4}\left(\frac{1}{s+1}\right) + \frac{3}{2}\left(\frac{s}{s^2+1^2}\right)$$

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$\therefore y(t) = \mathcal{L}^{-1}\{Y(s)\} = \boxed{\frac{-1}{4} e^{t} - \frac{1}{4} e^{-t} + \frac{3}{2}\cos t}$ or $\boxed{\frac{-1}{2}\sinh t + \frac{3}{2}\cos t}$

- **9. (30 points)** Find the general solution of the linear system, sketch the graph and describe the behavior as $t \to \infty$.

$$\begin{cases} x' &= 2x - 5y \\ y' &= x - 2y. \end{cases} \Rightarrow X' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} X$$

$$\begin{vmatrix} (2-\lambda) & -5 \\ 1 & (-2-\lambda) \end{vmatrix} = (2-\lambda)(-2-\lambda)+5 = -4-\cancel{2\lambda}+\cancel{2\lambda}+\lambda^2+5 = \lambda^2+1=0 \Rightarrow \lambda^2=-1 \Rightarrow \lambda = \pm i$$

for $\lambda = i$

$$\left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & (-2-i) & 0 \end{array}\right] \Rightarrow \begin{array}{l} u+(-2-i)v=0 \\ \therefore u = -(-2-i)v \\ \therefore u = (2+i)v \end{array} \Rightarrow \boxed{\xi = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}}$$

$$\therefore \xi e^{it} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}(\cos t + i\sin t) = \begin{bmatrix} 2\cos t + 2i\sin t + i\cos t - \sin t \\ \cos t + i\sin t \end{bmatrix} = \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + i\begin{bmatrix} 2\sin t + \cos t \\ \sin t \end{bmatrix}$$

$$\boxed{x(t) = c_1\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}}$$

as $t \to \infty$, depending on the initial condition, x follows the concentric circle path

$\boxed{\text{Center}}$

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$v = \frac{4}{2}u = 2u$

- **10. (30 points)** Solve the system of equations

$$\begin{cases} x_1' &= 4x_1 - 2x_2 \\ x_2' &= 8x_1 - 4x_2 \end{cases} \quad X' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} X$$

$$\begin{vmatrix} (4-\lambda) & -2 \\ 8 & (-4-\lambda) \end{vmatrix} = (4-\lambda)(-4-\lambda)+16 = -\cancel{16}-\cancel{4\lambda}+\cancel{4\lambda}+\lambda^2+\cancel{16} = \lambda^2 = 0 \Rightarrow \lambda_{1,2} = 0$$

$v=0$
0
$2v+1=0$
$2v=-1$

$v=0$
$4u=1$
$u=1/4$

for $\lambda = 0$,

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 8 & -4 & 0 \end{array}\right] \Rightarrow 4u = 2v \Rightarrow \xi = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow x^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0t} = \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix} = x^{(1)}}$$

$$\boxed{x^{(2)} = \xi t + \eta} \Rightarrow (A - \lambda I)\eta = \xi$$

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{array}{l} 4u-2v=1 \\ 4u = 1+2v \end{array} \Rightarrow \boxed{\eta = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}} \Rightarrow \boxed{x^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}}$$

$$\therefore x(t) = c_1 x^{(1)} + c_2 x^{(2)}$$

$$\boxed{x(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}\right)}$$

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