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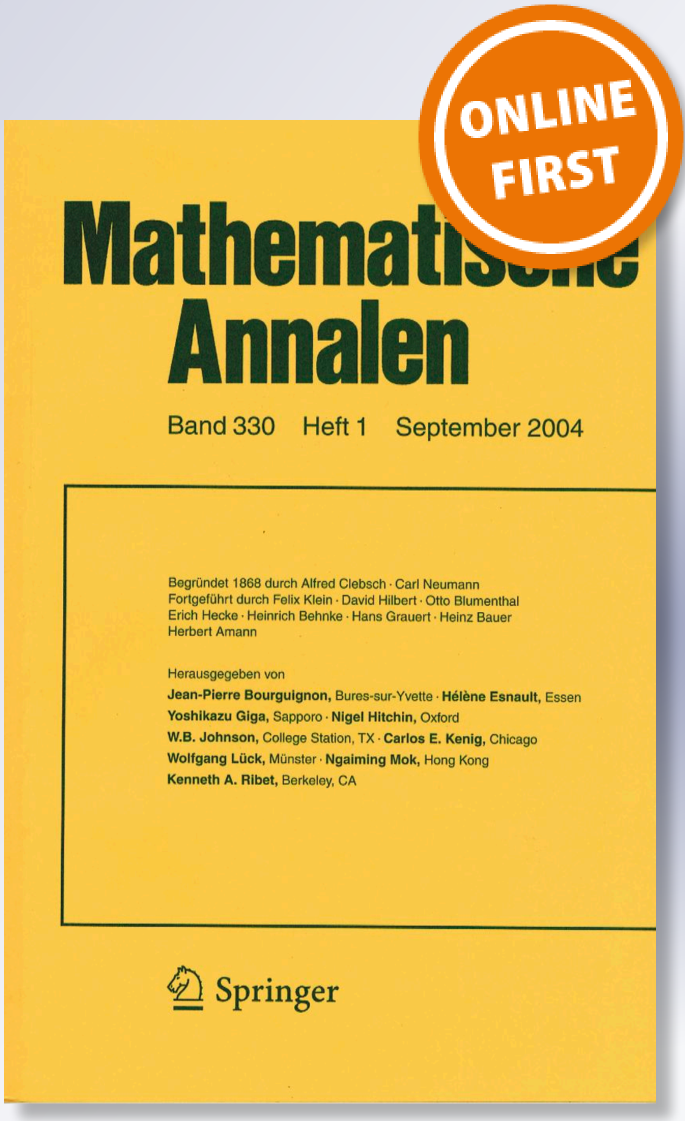

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When is the energy of the 1D damped Klein-Gordon equation decaying?

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Abstract
We study time decay of the energy for the one dimensional damped Klein-Gordon equation. We give an explicit necessary and sufficient condition on the continuous damping function $\gamma \geq 0$ for which the energy

$$E(t) = \int_{-\infty}^{\infty} |u_x|^2 + |u|^2 + |u_t|^2 dx$$

decays, whenever $(u(0), u_t(0)) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$.

Mathematics Subject Classification 35B35 · 35B40 · 35G30

1 Introduction

The main object of study is the following damped Klein-Gordon equation

$$u_{tt} + \gamma(x)u_t - u_{xx} + u = 0. \quad (x, t) \in \mathbb{R} \times \mathbb{R} \tag{1}$$

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where $\gamma(x)u_t$ represents a damping force proportional to the velocity u_t. This is a standard model in the theory. In the case $\gamma(x) = const.$, one can easily see that the energy function

$$E(u) = \int_{-\infty}^{\infty} |u_x|^2 + |u|^2 + |u_t|^2 dx,$$

has an exponential decay as $t \to \infty$. Thus a natural question to ask is the following: under what conditions on $\gamma(x) \geq 0$, one can still guarantee such exponential (or slower algebraic) decay. This question was intensely researched in the last ten years. We present a brief (and definitely incomplete) overview of the recent results.

In this direction, Burq and Joly have recently proved in [6] exponential rate of decay of the semigroup under the geometric control condition (GCC) in a sense that there exist $T, \epsilon > 0$, such that $\int_0^T \gamma(x(t))dt \geq \epsilon$ along every straight line unit speed trajectory thus extending the previous work of Bardos, Lebeau, Rauch, and Taylor [2,3,14] of compact manifold to the whole space $\mathbb{R}^{\mathbb{N}}$. Notice that in [6] the authors also require additional uniform continuity requirement on the damping coefficient γ in order to use pseudo-differential calculus. The authors also provide counter examples [6](see fig 1c) where exponential decay is expected but regularity hypothesis of GCC failed badly. However this is not in the case of compact manifold where this assumption is automatically true.

In the absence of geometric control condition, the same authors of [6] also provide a weaker hypothesis, namely network control condition where the damping coefficient $\gamma(x)$ is strictly positive on a family of balls whose dilates cover $\mathbb{R}^N$ under which the solution of damped wave equation decays with logarithmic rate (still without loss of regularity). For a fixed periodic damping, Wunsch proved in [16] that without any geometric condition there is at least a polynomial (certainly not optimal) decay (with loss of regularity).

One can observe that in the case of compact manifold (see [1,5,12,15] and references therein) the decay rate of the semigroup of damped wave equation highly depends on the way the damping coefficient γ vanishes. Several sharp result are obtained in different settings. One should expect the same in the case of non compact setting. However, it is not clear in this case what is the optimal form of a damping coefficient which will ensure that one can expect exponential (or algebraic) energy decay to the solution of (1). The purpose of this paper is to find optimal conditions on the damping coefficient γ under which the exponential decay holds. In fact, we are able to provide a simple to verify necessary and sufficient condition for such an exponential decay.

1.1 Semigroup representations and some previous results

We begin by recasting (1) as an abstract Cauchy problem. Define $U = (u, u_t)^\top$, then Eq. (1) can be written as a dynamical system in the following form

$$U_t = \mathcal{A}U, \quad \mathcal{A} = \begin{pmatrix} 0 & I \\ \partial_x^2 - 1 & -\gamma(x) \end{pmatrix} \tag{2}$$

The operator $\mathcal{A}$ is defined on a Hilbert space $\mathcal{H} = H^1(\mathbb{R}) \times L^2(\mathbb{R})$, with domain $H^2(\mathbb{R}) \times H^1(\mathbb{R})$. It is well-known that if the damping $\gamma \geq 0$ is bounded, this defines a semigroup $T(t)$ (see [11]). In fact, $T(t)$ is a semigroup of contractions.

The following is the main result of the article.

Theorem 1 *Assume $\gamma : \mathbb{R} \to \mathbb{R}$, with $\gamma \geq 0$ is a continuous and bounded function. The following statements are equivalent*

(i)

$$\liminf_{N\to\infty} \inf_{y\in\mathbb{R}} \frac{1}{N} \int_y^{y+N} \gamma(z)dz > 0. \tag{3}$$

(ii) $1 \in \rho(\mathcal{A})$ *and there exists* $\lambda_0 > 0$, *so that*

$$\|e^{t\mathcal{A}}(1-\mathcal{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2} \leq Ce^{-\lambda_0 t}.$$

Equivalently,

$$\|(u(t), u_t(t))\|_{H^1\times L^2} \leq Ce^{-\lambda_0 t}\|(u(0), u_t(0))\|_{H^2\times H^1}$$

whenever $(u(0), u_t(0)) \in H^2 \times H^1$.

(iii) $\lim_{t\to\infty} \|e^{t\mathcal{A}}\|_{H^2\times H^1\to H^1\times L^2} = 0$.

(iv) *For the semigroup generated by* (1), $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

The proof of Theorem (1) is based on the semigroup technique used in [5,8,10,16], in which rather than estimating the norm of the solution directly, we use a result obtained by Gearhart–Prüss, [7,13]. We use Theorem 2 which is a formulation given by Theorem 3 of [9] . More concretely, this result makes it possible to deduce exponential rate of decay of the energy of the solution by uniformly estimating the norm of the resolvent $(\mathcal{A} - \lambda I)^{-1}$ of the generator of the semigroup on the imaginary axis. Some additional remarks are in order.

Remark 1. The condition (3), in the context of γ bounded is equivalent to

$$\liminf_{N\to\infty} \inf_{y\in\mathbb{R}} \frac{1}{N} \int_y^{y+N} \gamma^p(z)dz > 0$$

for any $p > 1$. This is a consequence of the Hölder's inequality

$$\frac{1}{N}\int_y^{y+N} \gamma(z)dz \leq \left(\frac{1}{N}\int_y^{y+N} \gamma^p(z)dz\right)^{\frac{1}{p}}$$
$$\leq \|\gamma\|_{L^\infty}^{\frac{p-1}{p}} \left(\frac{1}{N}\int_y^{y+N} \gamma(z)dz\right)^{\frac{1}{p}}$$

2. The implication $(i) \Rightarrow (ii)$ above is of course trivial. The equivalence, namely the fact that $(iii) \Rightarrow (ii)$, means that as long as a solution starting with an initial data

in $H^2 \times H^1$ goes to zero in the energy norm $H^1 \times L^2$, then this convergence must be exponential. In particular, this implies that algebraic convergence is impossible. However, exponential convergence is possible. This is of course in sharp contrast with the higher dimensional case, where algebraic convergence is possible [6,16].

3. The equivalence $(iii) \Leftrightarrow (iv)$ is a particular case for the bounded semigroup[1] of the damped wave equation (1), of a more general theorem of Batty–Borichev–Tomilov, [4], Theorem 1.4 . See Theorem 3 below as well as the Corollary 2.

The paper is set out as follows. In Sect. 2 we show that our problem is well posed in the sense of C_0-semigroups and we describe the spectrum of the infinitesimal generator. In Sect. 3 we turn to compute the resolvent bound of the semigroup. The method we use here to find the resolvent bound is very functional analytical. However, this is the most technical part of the paper. At the end of the section, we apply the Gearhart–Prüss Theorem 2 to deduce from the resolvent bound an estimate for the rate of energy decay of smooth solutions.

2 Preliminaries and Notations

In order to fix notations, the Fourier transform will henceforth take the form

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx, \quad f(x) = (2\pi)^{-1}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}d\xi.$$

Henceforth, the constant C will change from line to line , but will always be independent of the spectral parameter. The constants C_δ and C_ϵ are different constant with dependence on δ and ϵ respectably. These constants also will change line to line throughout the presentation.

Proposition 1 *Let $\gamma \geq 0$ be a bounded function. Then, we have*

$$\|T(t)\|_{\mathcal{H}\to\mathcal{H}} \leq 1 \qquad \forall \ t \geq 0.$$

Proof All we need for the proof is to take a sufficiently smooth and decaying initial data for (1), consider its solution at a later time and take a dot product with $u_t \in L^2(\mathbb{R})$. We obtain,

$$\partial_t(|u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2) + \int \gamma|u_t|^2dx = 0.$$

It follows that the energy function $E(t) = \|u_t(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \|u_x(t)\|_{L^2}^2$ is decaying with time, hence $E(t) \leq E(0)$, or equivalently $\|(u(t), u_t(t))\|_{\mathcal{H}} \leq \|(u(0), u_t(0))\|_{\mathcal{H}}$. □

Next, we have the following interesting corollary.

[1] See Proposition 1.

Corollary 1 *Let $\gamma \geq 0$ be a continuous function, so that (3) does not hold. That is*

$$\liminf_{N\to\infty} \inf_{y\in\mathbb{R}} \frac{1}{N} \int_y^{y+N} \gamma(z)dz = 0$$

Then, $\|e^{t\mathcal{A}}\|_{H^1\times L^2\to H^1\times L^2} = 1$, for all $t \geq 0$.

Proof By Proposition 1, for $T(t) = e^{t\mathcal{A}}$, we have

$$\|T(t)\|_{H^1\times L^2\to H^1\times L^2} \leq 1$$

and $T(0) = Id$. Clearly $\|T(0)\| = 1$. Assume for a contradiction, that for some $t_0 > 0$,

$$\|T(t_0)\|_{H^1\times L^2\to H^1\times L^2} = q < 1.$$

From the equivalent condition (iii) of Theorem 1 above, it follows that

$$\limsup_{t\to\infty} \|T(t)(1-\mathcal{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2} \geq c_0 > 0.$$

Say, $t_n \to \infty$, so that

$$\|T(t_n)(1-\mathcal{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2} \geq \frac{c_0}{2}.$$

Now,

$$\begin{aligned}\frac{c_0}{2} &\leq \|T(t_n)(1-\mathcal{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2}\\ &\leq \|T(t_n)\|_{H^1\times L^2\to H^1\times L^2}\|(1-\mathcal{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2}\\ &\leq q^{[\frac{t_n}{t_0}]}\|(1-\mathcal{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2}.\end{aligned}$$

Since clearly, $\lim_n q^{[\frac{t_n}{t_0}]} = 0$, this is a contradiction. □

The following result will be one of the main technical tools that allows us to deduce exponential decay from estimates on the resolvent.

Theorem 2 (Gearhart–Prüss) *Let $e^{t\mathcal{A}}$ be a C_0-semigroup in a Hilbert space X and assume that there exists a positive constant $M > 0$ such that $\|e^{t\mathcal{A}}\| \leq M$ for all $t \geq 0$. Let $\mu \in \rho(\mathcal{A})$, then the following are equivalent.*

(i) *There exists $\lambda_0 > 0$ and C, so that*

$$\|T(t)(\mu-\mathcal{A})^{-1}\|_{B(X)} \leq Ce^{-\lambda_0 t}$$

(ii) $i\mathbb{R} \subset \rho(\mathcal{A})$ *and*

$$\sup_{s\in\mathbb{R}} \|(\mathcal{A} - isI)^{-1}\|_{B(X)} < +\infty.$$

Another result, which will be useful for us is the following.

Theorem 3 (Batty–Borichev–Tomilov, [4], Theorem 1.4) *Let $e^{t\mathcal{A}}$ be a bounded C_0-semigroup in a Banach space X. Then for $\mu \in \rho(\mathcal{A})$, the following are equivalent*

(i) $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$
(ii) $\lim_{t\to\infty} \|T(t)(\mu - \mathcal{A})^{-1}\|_{B(X)} = 0.$

Note that in the case of the damped wave equation semigroup (2), say with $\mu = 1$, $(1 - \mathcal{A})^{-1} : H^1 \times L^2 \to H^2 \times H^1$ and this map is onto. Thus, an application of Theorem 3 to this particular case yields the following.

Corollary 2 *For the semigroup $T(t)$ of damped wave Eq.* (2)*, the following are equivalent*

(i) $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$
(ii) $\lim_{t\to\infty} \|T(t)\|_{H^2\times H^1\to H^1\times L^2} = 0.$

2.1 Spectrum of $\mathcal{A}$

We begin by (formally) computing the resolvent of the operator $\mathcal{A}$ as follows: Let $u = (u_1, u_2)^\top$ and $f = (f_1, f_2)^\top$ then $(isI - \mathcal{A})u = f$ is equivalent to

$$\begin{aligned} isu_1 - u_2 &= f_1 \\ (-\partial_x^2 + 1)u_1 + (is + \gamma(x))u_2 &= f_2 \end{aligned}$$

or

$$\begin{aligned} u_1 &= (-\partial_x^2 + 1 + is\gamma(x) - s^2)^{-1}\left((is + \gamma(x))f_1 + f_2\right) \\ u_2 &= (-\partial_x^2 + 1 + is\gamma(x) - s^2)^{-1}\left(isf_2 - (-\partial_x^2 + 1)f_1\right) \end{aligned}$$

Hence, if we introduce the resolvent operator $R(is) := (-\partial_x^2 + 1 + is\gamma(x) - s^2)^{-1}$, then resolvent operator of $\mathcal{A}$ is denoted by $\mathcal{R}(is, \mathcal{A})$ and is given by

$$\mathcal{R}(is, \mathcal{A}) = \begin{pmatrix} R(is)(is + \gamma(x)) & R(is) \\ R(is)(is)(\gamma(x) + is) - I & R(is)(is) \end{pmatrix}. \tag{4}$$

From this, we see that in order to study $\mathcal{R}(is, \mathcal{A})$ it suffices to understand $R(is)$. In fact, by inspecting the form of the resolvent (4) in a way similar to [16], we have the following.

Lemma 1 *The following are equivalent*

(i) $\mathrm{i}s \in \rho(\mathcal{A})$
(ii) $0 \in \rho(-\partial_x^2 + 1 + is\gamma(x) - s^2)$, *that is*

$$R(is) = (-\partial_x^2 + 1 + is\gamma(x) - s^2)^{-1} : L^2 \to H^2.$$

In fact, $(\mathrm{i}s)$ *is an eigenvalue of* $\mathcal{A}$ *if and only if* 0 *is an eigenvalue of* $A_s :=$ $(-\partial_x^2 + 1 + \mathrm{i}s\gamma(x) - s^2)$.

Note: In this lemma, we consider s fixed. In particular, we are not concerned with the behavior of the various norms as $|s| \to \infty$. This is a much more subtle issue, that we will deal with later.

According to Lemma 1, the set $\sigma(\mathcal{A}) \cap i\mathbb{R}$ can be characterized as those is, $s \in \mathbb{R}$, for which there exists $g_n \in H^2(\mathbb{R})$ with $\|g_n\|_{H^2} = 1$, so that

$$\lim_n \|A_s g_n\|_{L^2} = 0.$$

The purely imaginary spectrum $\sigma(\mathcal{A}) \cap i\mathbb{R}$ (if any!), naturally consists of two subsets—eigenvalues and the rest, which we call essential spectrum.[2] Namely, is is an eigenvalue, if there exists $g_s \neq 0$, $g_s \in H^2(\mathbb{R})$, so that $A_s g_s = 0$.

Proposition 2 *Let* $\gamma \geq 0$, $\gamma \neq 0$ *be a continuous function. Then,*

(i) $\mathcal{A}$ *has no purely imaginary eigenvalues.*
(ii) $i \in \sigma(\mathcal{A})$ *if and only if* $\sigma(\mathcal{A}) \supseteq \{i\lambda, \lambda \in \mathbb{R} : |\lambda| \geq 1\}$.

Finally, if there is $\delta > 0$, *so that* $\gamma(x) \geq \delta > 0$, *then* $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

Proof We show that there are no eigenvalues. First, we rule out the case $s = 0$. For $s = 0$, by Lemma 1, 0 will be an eigenvalue of $(-\partial_x^2 + 1)$. If so, there exist $g \neq 0$ such that $(-\partial_x^2 + 1)g = 0$, which is impossible. Take a dot product with g to conclude $\|g'\|_{L^2}^2 + \|g\|_{L^2}^2 = 0$ or $g = 0$.

Next, assume that $s \neq 0$ and there is an eigenvalue is of $\mathcal{A}$. Again by Lemma 1, 0 will be an eigenvalue of $(-\partial_x^2 + 1 + is\gamma - s^2)$. Let $f = f_1 + if_2$, $f \neq 0$ be the corresponding eigenfunction of eigenvalue 0. Then, taking real and imaginary part of the equation $(-\partial_x^2 + 1 + is\gamma - s^2)f = 0$, we obtain

$$\left| \begin{array}{l} (-\partial_x^2 + (1 - s^2))f_1 - s\gamma f_2 = 0, \\ (-\partial_x^2 + (1 - s^2))f_2 + s\gamma f_1 = 0. \end{array} \right.$$

Taking dot products with f_2 and f_1 respectively and subtracting, we obtain

$$\int_{\mathbb{R}} \gamma(x)(f_1^2 + f_2^2)dx = 0. \tag{5}$$

[2] Here, we depart from the usual definition, where eigenvalues of infinite multiplicities are considered as part of the essential spectrum. We will see though, that since eigenvalues do not appear in our setup, at least on the important set $\sigma(\mathcal{A}) \cap i\mathbb{R}$, this is not consequential.

Recall $\gamma \geq 0$. Since $\gamma \neq 0$, let (a, b) be an interval on which $\gamma(x) > 0$. Then, (5) implies that $f_1(x) = f_2(x) = 0$ for $x \in (a, b)$. By the uniqueness theorem for second order ODE's, $f_1 = f_2 = 0$ for the intervals $(-\infty, a)$, (b, ∞), so $f_1 = f_2 = 0$, contradiction.

Clearly, if $\sigma(\mathcal{A}) \supseteq \{i\lambda, \lambda \in \mathbb{R} : |\lambda| \geq 1\}$, it follows that $i \in \sigma(\mathcal{A})$. Now, assume that $i \in \sigma(\mathcal{A})$. It follows that for a sequence g_n with $\|g_n\|_{H^2} = 1$, we have

$$(-\partial_x^2 + i\gamma)g_n = f_n,$$

where $\|f_n\|_{L^2} \to 0$. Taking dot product with g_n and then imaginary part yields

$$0 \leq \int \gamma |g_n|^2 = \Im\langle f_n, g_n\rangle \leq \|f_n\|_{L^2}\|g_n\|_{L^2} \to 0.$$

It follows that $\|\sqrt{\gamma} g_n\|_{L^2}^2 = \int \gamma |g_n|^2 \to 0$. Let $\tilde{f}_n := f_n - i\gamma g_n$. Clearly, $\|\tilde{f}_n\|_{L^2} \to 0$ and $-g_n'' = \tilde{f}_n$. Note that since $\|g_n''\|_{L^2} = \|\tilde{f}_n\|_{L^2} \to 0$, we have

$$1 = \|g_n\|_{H^2} \sim \|g_n''\|_{L^2} + \|g_n\|_{L^2},$$

whence $\liminf_n \|g_n\|_{L^2} > 0$.

Now, let $s \in \mathbb{R}$ such that $|s| > 1$. Consider $\mu := \sqrt{s^2 - 1} > 0$. Introduce $u_n := e^{i\mu x} g_n$, so $\liminf_n \|u_n\|_{L^2} = \liminf_n \|g_n\|_{L^2} > 0$. Compute

$$A_s u_n = (-\partial_x^2 + is\gamma - \mu^2)(g_n e^{i\mu x}) = e^{i\mu x}(-g_n'' - 2i\mu g_n' + is\gamma g_n)$$

We have

$$\|A_s u_n\|_{L^2} \leq \|g_n''\|_{L^2} + 2\mu\|g_n'\|_{L^2} + |s|\|\gamma g_n\|_{L^2}$$

Since all of the quantities on the right were shown to converge to zero, it follows that $\lim_n \|A_s u_n\|_{L^2} = 0$, while $\liminf_n \|u_n\|_{L^2} > 0$. Thus, $is \in \sigma(\mathcal{A})$ for all $s \in \mathbb{R}$ such that $|s| > 1$.

For the last part, assume that $\gamma(x) \geq \delta$ and yet is is in $\sigma(\mathcal{A})$. We saw $s = 0$ is not an option. So, $s \neq 0$. That is

$$(-\partial_x^2 + 1 - s^2 + is\gamma)g_n = f_n. \tag{6}$$

Taking dot product with g_n and then imaginary parts yields

$$|s| \int \gamma |g_n|^2 dx \leq |\langle f_n, g_n\rangle| \leq \|f_n\|\|g_n\|.$$

It follows that

$$\delta|s| \int |g_n|^2 dx \leq \|f_n\|\|g_n\| \to 0,$$

so $\|g_n\| \to 0$. But from the Eq. (6),

$$\|g_n''\|_{L^2} \leq C(|s^2 - 1|\|g_n\| + \|g_n\| + \|f_n\|) \to 0.$$

So, it follows that $\|g_n\|_{H^2} \to 0$, a contradiction. $\square$

We now provide a sufficient condition for $\sigma(\mathcal{A}) \cap i\mathbb{R} \neq \emptyset$, which turns out, in a roundabout way, to be necessary as well.

Proposition 3 *Let $\gamma \geq 0$ be a bounded and continuous function, not identically zero. Assume that (3) does not hold, that is*

$$\liminf_{N\to\infty} \inf_{y\in\mathbb{R}} \frac{1}{N}\int_y^{y+N} \gamma(z)dz = 0. \tag{7}$$

Then, $\sigma(\mathcal{A}) \supseteq \{i\lambda, \lambda \in \mathbb{R} : |\lambda| \geq 1\}$.

Proof By Proposition 2, it suffices to check that $i \in \sigma(\mathcal{A})$. It will be an element of the essential spectrum, since as we have shown there are no eigenvalues. By (7), we can find a sequences $y_j \in \mathbb{R}$, $N_j \to \infty$, so that

$$\lim_j \frac{1}{N_j}\int_{y_j}^{y_j+N_j} \gamma(z)dz = 0.$$

Consider $\Psi \neq 0 \in C_0^\infty(\mathbb{R})$ with $0 \leq \Psi(z) \leq 1$, so that $\Psi(z) = 0$ for $z < 0$ and $\Psi(z) = 0$, $z > 1$. Let $\epsilon_j := N_j^{-1} \to 0$ and take u_j so that

$$u_j(x) := \sqrt{\epsilon_j}\Psi(\epsilon_j(x - y_j)).$$

Clearly, $\|u_j''\|_{L^2} \to 0$, while $\|u_j\|_{L^2} = \|\Psi\|_{L^2} = O(1)$.

Recall $A_s = (-\partial_x^2 + 1 + is\gamma - s^2)$. We compute the norm of A_s for $s = 1$ as follows

$$\|A_1(u_j)\|_{L^2} = \|(-\partial_x^2 + i\gamma)u_j\|_{L^2} \leq C(\|u_j''\|_{L^2} + \|\gamma u_j\|_{L^2}).$$

We have already seen $\|u_j''\|_{L^2} \to 0$. For the other term,

$$\|\gamma u_j\|_{L^2}^2 \leq \|\gamma\|_{L^\infty}\epsilon_j \int \gamma(x)|\Psi(\epsilon_j(x - y_j))|^2 dx \leq \|\gamma\|_{L^\infty} \frac{1}{N_j}\int_{y_j}^{y_j+N_j} \gamma(z)dz.$$

It follows that $\lim_j \|\gamma u_j\|_{L^2} = 0$, whence Proposition 3 is established. $\square$

2.2 The analysis of control hypothesis

Let us analyze (3) in a more quantitative way. It means that there exists κ_γ and N_γ, so that for all $N > N_\gamma$ and for all $y \in \mathbb{R}$, we have

$$\frac{1}{N}\int_y^{y+N} \gamma(z)dz \geq \kappa_\gamma. \tag{8}$$

We have the following technical lemma, which will be useful later on.

Lemma 2 *Let $\tilde{\gamma} \geq \gamma \geq 0$ are continuous functions, so that γ satisfies (8). Then, for every $x, y \in \mathbb{R}$*

$$exp\left(-\int_{\min(x,y)}^{\max(x,y)} \tilde{\gamma}(z)dz\right) \leq e^{2N_\gamma \kappa_\gamma} e^{-\kappa_\gamma |x-y|}. \tag{9}$$

Proof Consider the case $0 \leq x < y$. Clearly, the case $x < y < 0$ follows by symmetry and then the case $x < 0 < y$ follows by applying the previous two cases to $x < 0 = y$ and $0 = x < y$.

We bound $\int_x^y \tilde{\gamma}(z)dz \geq 0$, if $y - x < N_\gamma$. When $y - x \geq N_\gamma$, we have by (8),

$$\int_x^y \tilde{\gamma}(z)dz \geq \kappa_\gamma(y-x).$$

Overall,

$$\begin{aligned} exp\left(-\int_{\min(x,y)}^{\max(x,y)} \tilde{\gamma}(z)dz\right) &\leq \begin{cases} 1 & y-x < N_\gamma \\ exp(-\kappa_\gamma(y-x)) & y-x \geq N_\gamma \end{cases} \\ &\leq e^{N_\gamma \kappa_\gamma} e^{-\kappa_\gamma(y-x)}. \end{aligned}$$

$\square$

3 Proof of Theorem 1

We stat with a technical result that gives bounds for the resolvent, under the appropriate condition (3). For all practical purposes, this is essentially the implication $(i) \Rightarrow (ii)$ of Theorem 1. For technical reasons, however, we will need to assume (as a preliminary step) that the spectrum does not intersect the imaginary access, that is $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$, so that the various quantities are well-defined. We remove this assumption later—in fact, we show, in a roundabout way, that indeed the property $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ follows from (3) alone.

3.1 The main technical step

Proposition 4 *Let $\gamma(x) \geq 0$ is a positive, continuous function, which satisfies (3) or equivalently (9). In addition, assume that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Accordingly, let $s \in \mathbb{R}$, $f \in L^2(\mathbb{R})$ and $u \in L^2(\mathbb{R})$ satisfy the resolvent equation*

$$(-\partial_x^2 + 1 + is\gamma(x) - s^2)u = f. \tag{10}$$

Then, for every $\delta > 0$, there is a constant $C_{\delta,\kappa,N}$, so that for all $s \in \mathbb{R}$ such that $|s|^2 \in [0, 1-\delta) \cup (1+\delta, \infty)$, we have

$$\|u\|_{L^2(\mathbb{R})} \leq \frac{C_{\delta,\kappa,N}}{1+|s|}\|f\|_{L^2(\mathbb{R})}, \tag{11}$$

where κ, N[3] are the quantitative bounds of γ from (8).

Proof We begin by pairing the Eq. (10) with u and taking the real part, we obtain by using Cauchy-Schwartz, for $s^2 < 1-\delta$

$$\|u'\|_{L^2}^2 + (1-s^2)\|u\|_{L^2}^2 = \Re\langle f, u\rangle \leq C_s\|f\|_{L^2(\mathbb{R})}^2 + \frac{1-s^2}{2}\|u\|_{L^2}^2.$$

It follows that

$$\|u\|_{H^1(\mathbb{R})}^2 \leq C_\delta\|f\|_{L^2(\mathbb{R})}^2,$$

Note that from this proof, the constant C_δ may blow up as $\delta \to 0$.

We now consider the case $|s|^2 \geq 1+\delta$. We assume that s is positive since the case for negative s can be obtained by changing s to $-s$.

Let $0 < \epsilon << 1$ be small enough, to be selected later and $\mu_s := \sqrt{s^2-1} \geq \sqrt{\delta} > 0$. We have $c_\delta|s| \leq \mu_s \leq C_\delta|s|$. Henceforth, all constants will implicitly depend on δ, but we will omit this dependence.

Introduce the operators $P_{\sim s}$, $P_{\sim -s}$ and $P_{\sim(s,-s)}$ through Fourier transform:

$$\begin{aligned}
\widehat{P_{\sim s}(f)}(\xi) &= \hat{f}(\xi)\,\psi\left(\frac{\xi-\mu_s}{\epsilon}\right)\\
\widehat{P_{\sim(-s)}(f)}(\xi) &= \hat{f}(\xi)\,\psi\left(\frac{\xi+\mu_s}{\epsilon}\right),\\
P_{\nsim(s,-s)}(f)(\xi) &= (Id - P_{\sim s} - P_{\sim(-s)})f.
\end{aligned}$$

Here $\psi \in C_0^\infty(\mathbb{R})$ is an even function,

$$\psi(z) = 1 \text{ for } |z| < 1 \text{ and } \psi(z) = 0,\ |z| > 2.$$

[3] The constants N and κ have subscript γ, however we will remove this in the rest of the presentation.

Further, we use the notation

$$P_{\sim s}u(x) := u_{\sim s}(x),\ \ P_{\nsim s}u(x) := u_{\nsim s}(x),\ \ P_{\nsim(s,-s)}(u(x)) := u_{\nsim(s,-s)}(x).$$

We will prove the proposition by projecting the Eq. (10) into the three regions for s using the above projections and estimating the norm of u on each.

Taking dot product of (10) with u and using the imaginary parts and Cauchy-Schwartz's inequality yields the following estimates

$$s\int_{\mathbb{R}}\gamma(x)|u|^2dx \le \|f\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}.$$

Thus, we can conclude

$$\|\sqrt{\gamma}u\|_{L^2} \le \epsilon\|u\|_{L^2} + C_\epsilon\frac{\|f\|}{s} \tag{12}$$

where C_ϵ is a constant which depends on ϵ.

Part I: apply $P_{\nsim(s,-s)}$ on both sides of the Eq. (10) to get

$$(-\partial_x^2)u_{\nsim(s,-s)}(x) - \mu_s^2u_{\nsim(s,-s)}(x) = -is(\gamma u)_{\nsim(s,-s)}(x) + f_{\nsim(s,-s)}(x).$$

Using Fourier transform and the fact that ξ is away from μ_s and $-\mu_s$, we get

$$\widehat{u}_{\nsim(s,-s)}(\xi) = \frac{-is}{(\xi^2-\mu_s^2)}\left(\widehat{(\gamma u)}_{\nsim(s,-s)}(\xi)\right) + \frac{1}{\xi^2-\mu_s^2}\widehat{f}_{\nsim(s,-s)}(\xi).$$

On the support of $\widehat{u}_{\nsim(s,-s)}(\xi)$, we clearly have $|\frac{-is}{(\xi^2-\mu_s^2)}| \le C$, for some constant C. This gives the following estimate,

$$\|u_{\nsim(s,-s)}\|_{L^2} \le C\left(\|(\gamma u)_{\nsim(s,-s)}\|_{L^2} + \frac{\|f_{\nsim(s,-s)}\|_{L^2}}{s}\right) \le C\left(\|\gamma u\|_{L^2} + \frac{\|f\|_{L^2}}{s}\right)$$

Then by (12), together with the fact that $\gamma \le C\sqrt{\gamma}$ a.e, we get

$$\|u_{\nsim(s,-s)}\|_{L^2} \le \epsilon\|u\|_{L^2} + C_\epsilon\frac{\|f\|_{L^2}}{s}. \tag{13}$$

Part II: apply $P_{\sim s}$ on both sides of the Eq. (10). Adding and subtracting $i\mu_s\gamma u_{\sim s}(x)$ we get

$$-\partial_x^2u_{\sim s}(x) + i\mu_s\gamma(x)u_{\sim s}(x) - \mu_s^2u_{\sim s}(x) = f_{\sim s}(x) - is(\gamma u)_{\sim s}(x) + i\mu_s\gamma u_{\sim s}(x)$$

Let $f = e^{i\mu_sx}F$ and $u = e^{i\mu_sx}U$ and observe that $P_{\sim s}(e^{i\mu_sx}g) = e^{i\mu_sx}P_{\sim 1}(g)$. We get

$$-\partial_x^2 U_{\sim 1}(x) - 2i\mu_s \frac{d}{dx} U_{\sim 1}(x) + i\mu_s \gamma U_{\sim 1}(x)$$
$$= F_{\sim 1}(x) - is(\gamma U)_{\sim 1} + i\mu_s \gamma U_{\sim 1}(x).$$

Hence, dividing by $-2i\mu_s$,

$$\frac{d}{dx}(U_{\sim 1}(x)) - \frac{\gamma(x)}{2} U_{\sim 1}(x) = \frac{i}{2\mu_s} \partial_x^2 U_{\sim 1}(x) + \frac{i}{2\mu_s} F_{\sim 1}(x) + \frac{s}{2\mu_s} (\gamma U)_{\sim 1}(x)$$
$$-\frac{1}{2}\gamma(x) U_{\sim 1}(x).$$

Using the integrating factor $e^{-\frac{1}{2}\int_0^x \gamma(y)dy}$, we solve in the form

$$U_{\sim 1}(x) = -\int_x^\infty e^{\frac{1}{2}\int_y^x \gamma(z)dz} G(y)\, dy = -T(G),$$

where $G = \frac{i}{2\mu_s}\partial_x^2 U_{\sim 1} + \frac{i}{2\mu_s} F_{\sim 1} + \frac{s}{2\mu_s}(\gamma U)_{\sim 1} - \frac{1}{2}\gamma U_{\sim 1}$ and T is an operator in the form

$$T(f)(x) = \int_x^\infty e^{\frac{1}{2}\int_y^x \gamma(z)dz} f(y) dy.$$

Note that by the bound (9), we have that

$$|T(f)(x)| \le \int_x^\infty e^{2N\kappa} e^{-\kappa|x-y|} |f(y)| dy,$$

whence

$$\|Tf\|_{L^2} \le \|e^{2N\kappa} e^{-\kappa|\cdot|}\|_{L^1} \|f\|_{L^2} = \frac{2e^{2N\kappa}}{\kappa} \|f\|_{L^2}.$$

In particular, the operator norm $\|T\|_{L^2 \to L^2}$ depends only on N, κ.

Now, since $U_{\sim 1}(x) = e^{-i\mu_s x} u_{\sim s}(x)$, rewrite

$$\gamma U_{\sim 1}(x) = e^{-i\mu_s x} \gamma u_{\sim s}(x) = e^{-i\mu_s x}((\gamma u)(x) - \gamma(x) u_{\sim -s}(x) - \gamma(x) u_{\not\sim (s,-s)}(x)).$$

Thus, introduce the effective right hand side

$$G_1 := \frac{i}{2\mu_s}\partial_x^2 U_{\sim 1} + \frac{i}{2\mu_s} F_{\sim 1} + \frac{s}{2\mu_s}(\gamma U)_{\sim 1} + e^{-i\mu_s x}(\gamma u - \gamma u_{\not\sim (s,-s)}),$$

so that $u_{\sim s}(x)$ and $u_{\sim -s}(x)$ are now in the relation

$$u_{\sim s}(x) - \frac{1}{2} e^{i\mu_s x} T(e^{-i\mu_s x} \gamma(x) u_{\sim -s}(x)) = e^{i\mu_s x} T(G_1). \tag{14}$$

Multiplying the last equation by $\sqrt{\gamma}$ and by introducing a new linear operator $T_s f := \frac{1}{2}e^{i\mu_s x}\sqrt{\gamma}T(e^{-i\mu_s x}\sqrt{\gamma} f)$, we can record the last relation as follows

$$\sqrt{\gamma}u_{\sim s} - T_s(\sqrt{\gamma}u_{\sim -s}) = e^{i\mu_s x}\sqrt{\gamma}T(G_1). \tag{15}$$

Similar arguments apply to $u_{\sim -s}$. More concretely, projecting $P_{\sim(-s)}$ on both sides to the Eq. (10), and adding $i\mu_s\gamma u_{\sim(-s)}$, we get

$$\begin{aligned} -\partial_x^2 u_{\sim(-s)}(x) + i\mu_s\gamma(x)u_{\sim(-s)}(x) - \mu_s^2 u_{\sim(-s)}(x) = -is(\gamma u)_{\sim(-s)}(x) \\ +i\mu_s\gamma u_{\sim(-s)}(x) + f_{\sim(-s)}(x). \end{aligned} \tag{16}$$

Letting now $f = e^{-i\mu_s x}\bar{F}$ and $u = e^{-i\mu_s x}\bar{U}$ and observing that

$$P_{\sim(-s)}(e^{-i\mu_s x}g) = e^{-i\mu_s x}P_{\sim 1}(g).$$

By (16), we obtain the equation

$$\begin{aligned} \frac{d}{dx}\bar{U}_{\sim 1}(x) + \frac{\gamma(x)}{2}\bar{U}_{\sim 1}(x) = -\frac{i}{2\mu_s}\partial_x^2\bar{U}_{\sim 1}(x) - \frac{s}{2\mu_s}(\gamma(x)\bar{U})_{\sim 1}(x) \\ +\frac{1}{2}\gamma(x)\bar{U}_{\sim 1}(x) - \frac{i}{2\mu_s}\bar{F}_{\sim 1}(x) \end{aligned}$$

With the help of the integrating factor $e^{\frac{1}{2}\int_0^x \gamma(z)dz}$, we solve the equation (by integrating from $-\infty$ to x) as follows

$$\bar{U}_{\sim 1}(x) = \int_{-\infty}^{x} e^{\frac{1}{2}\int_x^y \gamma(z)dz}D(y)dy = T^*(D), \tag{17}$$

where the right hand side is $D = -\frac{i}{2\mu_s}\partial_x^2\bar{U}_{\sim 1} - \frac{s}{2\mu_s}(\gamma(x)\bar{U})_{\sim 1} + \frac{1}{2}\gamma\bar{U}_{\sim 1} - \frac{i}{2\mu_s}\bar{F}_{\sim 1}$. Again,

$$\gamma\bar{U}_{\sim 1}(x) = e^{i\mu_s x}\gamma u_{\sim -s}(x) = e^{i\mu_s x}(\gamma(x)u(x) - \gamma(x)u_{\sim s}(x) - \gamma(x)u_{\nsim(s,-s)}(x))$$

The effective right hand side becomes

$$D_1 := -\frac{i}{2\mu_s}\partial_x^2\bar{U}_{\sim 1} - \frac{s}{2\mu_s}(\gamma(x)\bar{U})_{\sim 1} - \frac{i}{2\mu_s}\bar{F}_{\sim 1} + \frac{1}{2}e^{i\mu_s x}\gamma u - \gamma u_{\nsim(s,-s)})$$

and we obtain the following reformulation of (17),

$$u_{\sim -s} + \frac{1}{2}e^{-i\mu_s x}T^*(e^{i\mu_s x}\gamma u_{\sim s}) = e^{-i\mu_s x}T^*(D_1). \tag{18}$$

Again, a multiplication with $\sqrt{\gamma}$ resolves (18) to

$$\sqrt{\gamma}u_{\sim -s} + T_s^*(\sqrt{\gamma}u_{\sim s}) = e^{-i\mu_s x}\sqrt{\gamma}T^*(D_1). \tag{19}$$

Where $T_s^* f := \frac{1}{2}e^{-i\mu_s x}\sqrt{\gamma}T^*(e^{i\mu_s x}\sqrt{\gamma}f)$

Combining (15) and (19) allows us to control $\sqrt{\gamma}u_{\sim\pm s}$ and ultimately $u_{\sim\pm s}$. Indeed,

$$\begin{aligned}\sqrt{\gamma}u_{\sim s} &= T_s(\sqrt{\gamma}u_{\sim -s}) + e^{i\mu_s x}\sqrt{\gamma}T(G_1)\\ &= T_s(-T_s^*(\sqrt{\gamma}u_{\sim s}) + e^{-i\mu_s x}\sqrt{\gamma}T^*(D_1)) + e^{i\mu_s x}\sqrt{\gamma}T(G_1)\end{aligned}$$

whence we obtain the following operator equation for $\sqrt{\gamma}u_{\sim s}$

$$(Id + T_s T_s^*)(\sqrt{\gamma}u_{\sim s}) = T_s(e^{-isx}\sqrt{\gamma}T^*(D_1)) + e^{isx}\sqrt{\gamma}T(G_1).$$

Since $(Id + T_s T_s^*)$ is a symmetric operator, $(Id + T_s T_s^*) \geq Id$, we have that it is invertible (in fact, $\|(Id + T_s T_s^*)^{-1}\|_{L^2\to L^2} \leq 1$),

$$\begin{aligned}\|\sqrt{\gamma}u_{\sim s}\|_{L^2} &\leq \|T_s(e^{-i\mu_s x}\sqrt{\gamma}T^*(D_1)) + e^{i\mu_s x}\sqrt{\gamma}T(G_1)\|_{L^2}\\ &\leq C(\|G_1\| + \|D_1\|),\end{aligned} \tag{20}$$

where in the last step, we have used that T, T_s, together with their adjoints are bounded on L^2, with bounds depending upon κ, N only.

So, it remains to find suitable bounds for $\|G_1\|_{L^2}$, $\|D_1\|_{L^2}$. We just provide the bounds for $\|G_1\|$, as the bounds for $\|D_1\|$ proceed in an identical way. Clearly,

$$\|\frac{i}{2\mu_s}F_{\sim 1}\|_{L^2} \leq \frac{C}{s}\|F\|_{L^2} = \frac{C}{s}\|f\|_{L^2}.$$

By Plancherel's

$$\begin{aligned}\|\frac{i}{2\mu_s}\partial_x^2 U_{\sim 1}\|_{L^2} &\leq \frac{C}{s}\|\xi^2\widehat{U_{\sim 1}}\|_{L^2} \leq \frac{C}{s}\|\xi^2\widehat{U}(\xi)\psi\left(\frac{\xi}{\epsilon}\right)\|_{L^2} \leq \frac{C\epsilon^2}{s}\|U_{\sim 1}\|_{L^2}\\ &\leq \epsilon\|u_{\sim s}\|_{L^2(\mathbb{R})},\end{aligned}$$

provided $C\sqrt{2}\epsilon \leq 1$. Next, by (12),

$$\|\frac{1}{2}(\gamma U)_{\sim 1} + e^{-i\mu_s x}\gamma u\|_{L^2} \leq \|\gamma U\|_{L^2} + \|\gamma u\| = 2\|\gamma u\|_{L^2} \leq \epsilon\|u\|_{L^2} + C_\epsilon\frac{\|f\|}{s}.$$

Finally, by (13),

$$\|\gamma u_{\nsim(s,-s)}\|_{L^2} \leq C\|u_{\nsim(s,-s)}\|_{L^2} \leq \epsilon\|u\|_{L^2} + C_\epsilon\frac{\|f\|}{s}.$$

Altogether, we obtain

$$\|G_1\| + \|D_1\| \leq C\epsilon\|u\|_{L^2} + C_\epsilon \frac{\|f\|}{s}. \tag{21}$$

Based on (20) and (21), we obtain the following estimate

$$\|\sqrt{\gamma}u_{\sim s}\|_{L^2} \leq C\epsilon\|u\|_{L^2} + C_\epsilon \frac{\|f\|}{s}.$$

Part III: clearly, the same estimate holds for $\|\sqrt{\gamma}u_{\sim -s}\|_{L^2}$.

In order to get estimates for $\|u_{\sim s}\|_{L^2}$, $\|u_{\sim -s}\|_{L^2}$, one can now use the forms (14) and (18), to deduce

$$\begin{aligned}\|u_{\sim s}\| + \|u_{\sim -s}\| &\leq C(\|\sqrt{\gamma}u_{\sim -s}\| + \|\sqrt{\gamma}u_{\sim s}\| + \|G_1\| + \|D_1\|)\\ &\leq C\epsilon\|u\|_{L^2} + C_\epsilon \frac{\|f\|}{s}.\end{aligned}$$

Finally, with some absolute constant C (and with some $C_\epsilon \sim \epsilon^{-1}$), we have

$$\|u\|_{L^2} \leq \|u_{\sim s}\| + \|u_{\sim -s}\| + \|u_{\nsim(s,-s)}\| \leq C\epsilon\|u\|_{L^2} + C_\epsilon \frac{\|f\|}{s}.$$

Clearly, a choice of ϵ such that $C\epsilon < \frac{1}{2}$, we obtain the desired bound (11). □

Next, we need an estimate for $L^2 \to H^1$ bounds of the resolvent $(-\partial_x^2 + 1 + is\gamma(x) - s^2)^{-1}$.

Proposition 5 *Let $\gamma \geq 0, \gamma \neq 0$ be a continuous function, that satisfies (9), with constants κ, N. In addition, assume $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.*

Let $\delta > 0$ and $|s|^2 \in (0, 1-\delta) \cup (1+\delta, \infty)$. Recalling $R(is) = (-\partial_x^2 + 1 + is\gamma(x) - s^2)^{-1}$, we have the following estimates

$$\begin{aligned}\|R(is)\|_{L^2 \to H^1} &\leq C_{\delta,\kappa,N}\\ \|R(is)\|_{H^{-1} \to L^2} &\leq C_{\delta,\kappa,N}\end{aligned} \tag{22}$$

As a consequence,

$$\|(is - \mathcal{A})^{-1}\|_{H^1 \times L^2 \to H^1 \times L^2} \leq C_{\delta,\kappa,N}. \tag{23}$$

Proof Let $u \in H^1(\mathbb{R})$ be the solution of (24)

$$(-\partial_x^2 + 1 + is\gamma(x) - s^2)u = f \tag{24}$$

for $f \in L^2$.

Taking dot product of (24) with u yields,

$$\langle -\partial_x^2 u, u\rangle + (1 - s^2)\langle u, u\rangle \leq \|f\|_{L^2}\|u\|_{L^2}$$

Hence,

$$\|u\|_{H^1}^2 \leq \|f\|_{L^2}\|u\|_{L^2} + (s^2-1)\|u\|_{L^2}^2$$

By Proposition 4, we get

$$\|u\|_{H^1}^2 \leq C_{\delta,\kappa,N}\|f\|_{L^2}\frac{\|f\|_{L^2(\mathbb{R}^2)}}{1+|s|} + C_{\delta,\kappa,N}\frac{(s^2-1)}{(1+|s|)^2}\|f\|_{L^2(\mathbb{R})}^2.$$

This proves

$$\|R(is)\|_{L^2\to H^1} \leq C_{\delta,\kappa,N}.$$

Hence by duality

$$\|R(is)\|_{H^{-1}\to L^2} \leq C_{\delta,\kappa,N}. \tag{25}$$

We now focus on (23), that is we show that the resolvent $R(is,\mathcal{A})$ of $\mathcal{A}$ is bounded in $H^1(\mathbb{R})\times L^2(\mathbb{R})$. We estimate the norm of $R(is,\mathcal{A})$ as follows,

$$\begin{aligned}\left\|R(is,\mathcal{A})\begin{pmatrix} f\\ g\end{pmatrix}\right\|_{H^1\times L^2} &= \|R(is)(is+\gamma(x))f\|_{H^1} + \|R(is)g\|_{H^1}\\ &+\|(R(is)(is)(\gamma(x)+is)-I)f\|_{L^2} + \|R(is)(is)g\|_{L^2}\end{aligned}$$

This implies that to estimate the norm of the resolvent operator $R(is,\mathcal{A})$ as an operator on $H^1\times L^2$, we need to obtain the following bounds

$$\begin{aligned}&\|R(is)\| = O(1) : L^2 \to H^1,\\ &\|R(is)(is+\gamma(x))\| = O(1) : H^1 \to H^1,\\ &\|sR(is)\| = O(1) : L^2 \to L^2,\\ &\|R(is)(is)(\gamma(x)+is)-I)\| = O(1) : H^1 \to L^2.\end{aligned}$$

The estimates for $sR(is)$ and $R(is)$ are in (11) and (22) respectively. In order to estimate

$$\|R(is)(is)[\gamma(x)+is)] - I\|_{H^1\to L^2},$$

we use that

$$R(is)(is)[\gamma(x)+is)] - I = R(is)(\partial_x^2 - 1),$$

and hence, combining (25) together with the fact that $\partial_x^2 : H^1 \to H^{-1}$ is continuous. For $f \in H^1(\mathbb{R})$, we have

$$\begin{aligned}\|(R(is)(is)[\gamma(x)+is)] - I)f\|_{L^2} = \|R(is)(\partial_x^2-1)f\|_{L^2} &\leq C\|(1-\partial_x^2)f\|_{H^{-1}}\\ &= C\|f\|_{H^1}\end{aligned}$$

This proves:

$$R(is)(is)(\gamma(x)+is)-I=O(1):H^1\to L^2 \tag{26}$$

It remains to estimate the norm of

$$R(is)(is+\gamma(x)):H^1\to H^1.$$

We rewrite the above operator as

$$R(is)(is+\gamma(x))=\frac{1}{is}[1+R(is)(\partial_x^2-1)] \tag{27}$$

If $f\in H^1$ and $\tilde{u}=R(is)(\partial_x^2-1)f\in H^1$, then

$$(-\partial_x^2+1+is\gamma(x)-s^2)\tilde{u}=(\partial_x^2-1)f\in H^{-1} \tag{28}$$

Pair the Eq. (28) with $\tilde{u}$ and take the real part to get,

$$\begin{aligned}\|\partial_x\tilde{u}\|_{L^2}^2-(s^2-1)\|\tilde{u}\|_{L^2}^2 &\le \|(-\partial_x^2+1)f\|_{H^{-1}}\|\tilde{u}\|_{H^1}\\ &\le \|f\|_{H^1}\|\tilde{u}\|_{H^1}.\end{aligned}$$

By Cauchy Schwartz inequality, we get

$$\|\tilde{u}\|_{H^1}^2\le 2(s^2-1)\|\tilde{u}\|_{L^2}^2+\|f\|_{H^1}^2. \tag{29}$$

Next, when we estimate the L^2- norm of $\tilde{u}=R(is)(\partial_x^2-1)f$, we used (26) to get

$$\|R(is)(\partial_x^2-1)f\|_{L^2}\le C\|f\|_{H^1} \tag{30}$$

Combining the estimates (29) and (30) proves that

$$R(is)(\partial_x^2-1)=O(|s|):H^1(\mathbb{R})\to H^1(\mathbb{R}).$$

Then by the Eq. (27), we have

$$\|R(is)(is+\gamma(x))f\|_{L^2}\le C\|f\|_{H^1}$$

Hence, $(is-\mathcal{A})^{-1}=O(1):H^1\times L^2\to H^1\times L^2$. □

3.2 Proof of Theorem 1: the implication (*i*) ⇒ (*ii*)

Take any $\gamma\ge 0$, a continuous, bounded and non-negative function, that satisfies (3). We would now like to prove exponential decay of the semigroup, as required in (ii) of Theorem 1. This is basically what Proposition 4 does, except that it in addition also assumes $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. This eventually turns out to be the case, but we have not proved that yet.

Instead, we proceed by an approximating argument. More specifically, fix $\epsilon > 0$ and consider $\gamma_\epsilon(x) := \gamma(x) + \epsilon$ and the corresponding operator $\mathcal{A}_\epsilon$. We immediately observe two things. First, since $\gamma_\epsilon \geq \epsilon > 0$, we have by Proposition 2, that $\sigma(\mathcal{A}_\epsilon) \cap i\mathbb{R} = \emptyset$. Second, γ_ϵ satisfies (8) with the constants κ, N of γ. Hence, γ_ϵ satisfies (9). Thus, we are ready to apply Proposition 4 to γ_ϵ. For a fixed $\delta > 0$ and $|s|^2 \in (0, 1-\delta) \cap (1+\delta, \infty)$, we have the estimate

$$\|(-\partial_x^2 + 1 + is(\gamma + \epsilon) - s^2)^{-1}\|_{L^2 \to L^2} \leq \frac{C_{\delta,\kappa,N}}{1+|s|}. \tag{31}$$

In particular, note that the above bound is *independent* upon the parameter $\epsilon > 0$. One can now take $\epsilon \to 0+$ in order to obtain the operator $(-\partial_x^2 + 1 + is\gamma - s^2)^{-1}$, together with the desired bounds on its $L^2 \to L^2$ operator norm. This could be justifies in at least two ways. One is to show that for a fixed s, the family $\{(-\partial_x^2 + 1 + is(\gamma+\epsilon) - s^2)^{-1}\}_{\epsilon>0}$ is Cauchy in $B(L^2)$, by using the resolvent identity. More or less equivalently, we can directly construct $(-\partial_x^2 + 1 + is\gamma - s^2)^{-1}$ by the resolvent identity and the Neumann theorem as follows

$$\begin{aligned}(-\partial_x^2 + 1 + is\gamma - s^2)^{-1} &:= (-\partial_x^2 + 1 + is(\gamma+\epsilon) - s^2)^{-1} \\ &\quad (Id - is\epsilon(-\partial_x^2 + 1 + is(\gamma+\epsilon) - s^2)^{-1})^{-1}.\end{aligned}$$

Indeed, in the formula above, the first inverse exists by (31), while the second inverse exists by von Neumann for all small enough ϵ, since

$$\|is\epsilon(-\partial_x^2 + 1 + is(\gamma+\epsilon) - s^2)^{-1}\|_{L^2 \to L^2} \leq C|s|\epsilon \frac{C_{\delta,\kappa,N}}{1+|s|} < \frac{1}{2}.$$

Now that we have constructed $(-\partial_x^2 + 1 + is\gamma - s^2)^{-1}$ for all $s \in \mathbb{R}$ such that $|s|^2 \in (0, 1-\delta) \cup (1+\delta, \infty)$, we deduce the bound

$$\|(-\partial_x^2 + 1 + is\gamma - s^2)^{-1}\|_{L^2 \to L^2} \leq \frac{C_{\delta,\kappa,N}}{1+|s|}, \tag{32}$$

by simply letting $\epsilon \to 0+$ in (31). In addition, this shows that $\{i\lambda : |\lambda| \neq 1\} \subset \rho(\mathcal{A})$, that is the whole imaginary line, with the possible exception of $\pm i$ are in the resolvent set of $\mathcal{A}$.

Now, we show that $\pm i$ also belong to the resolvent set of $\mathcal{A}$. Indeed, otherwise, we will have by Proposition 2, that $\sigma(\mathcal{A}) \supset \{i\lambda : |\lambda| > 1\}$, which is a contradiction. Thus, we have established that $\pm i \in \rho(\mathcal{A})$ or

$$\|(-\partial_x^2 \pm i\gamma)^{-1}\|_{L^2 \to L^2} \leq C.$$

Next, we show that (32) holds in a neighborhood of $|s| = 1$ as well. We have by the resolvent identity

$$(-\partial_x^2 + 1 + is\gamma - s^2)^{-1} - (-\partial_x^2 + i\gamma)^{-1} \\ = (-\partial_x^2 + 1 + is\gamma - s^2)^{-1}[s^2 - 1 + i\gamma(1-s)](-\partial_x^2 + i\gamma)^{-1},$$

whence we can represent

$$(-\partial_x^2 + 1 + is\gamma - s^2)^{-1} \\ = (-\partial_x^2 + i\gamma)^{-1}(Id - (s-1)(s+1-i\gamma)(-\partial_x^2 + i\gamma)^{-1})^{-1}.$$

Clearly, for $s \in \mathbb{R}$ with $|s-1| << 1$, say $(10+\|\gamma\|_{L^\infty})|s-1|\|(-\partial_x^2+i\gamma)^{-1}\|_{L^2\to L^2} \leq \frac{1}{2}$, the right-hand side is a well-defined operator and in addition

$$\|(-\partial_x^2 + 1 + is\gamma - s^2)^{-1}\|_{L^2\to L^2} \leq 2\|(-\partial_x^2 + i\gamma)^{-1}\|_{L^2\to L^2}.$$

Thus, $s \to \|(-\partial_x^2 + 1 + is\gamma - s^2)^{-1}\|_{L^2\to L^2}$ is bounded in a neighborhood of $s = 1$ and similarly, in a neighborhood of $s = -1$. In the same fashion as in Proposition 5, we conclude that

$$\sup_{s\in\mathbb{R}} \|(is - \mathcal{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2} \leq C < \infty.$$

By the Gearhart–Prüss theorem, $\|T(t)(1-\mathcal{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2} \leq Ce^{-\lambda_0 t}$, for some $\lambda_0 > 0$. Since, $(1-\mathcal{A})^{-1} : H^1 \times L^2 \to H^2 \times H^1$ and it is onto, we conclude that

$$\|Tt)g\|_{H^1\times L^2} \leq Ce^{-\lambda_0 t}\|g\|_{H^2\times H^1},$$

as stated.

Next, the implication $(ii) \Rightarrow (iii)$ is of course trivial. The equivalence of (iii) and (iv) is the essence of Theorem 3, see also Corollary 2. Finally, the implication $(iv) \Rightarrow (i)$ is contained in Proposition 3. This finishes the proof of Theorem 1.

References

1. Anantharaman, N., Léautaud, M.: Sharp polynomial decay rates for the damped wave equation on the torus. Anal. PDE **7**(1), 159–214 (2014). **(With an appendix by Stéphane Nonnenmacher)**
2. Bardos, C., Lebeau, G., Rauch, J.: Un exemple dutilisation des notions de propagation pour le contrôle et la stabilisation de problemes hyperboliques. Rend. Sem. Mat. Univ. Politec. Torino pp. 11–31 (1988)
3. Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim. **30**(5), 1024–1065 (1992)
4. Batty, C., Borichev, A., Tomilov, Y.: L^p-tauberian theorems and L^p-rates for energy decay. J. Functional Anal. **270**(3), 1153–1201 (2016)
5. Burq, N., Hitrik, M.: Energy decay for damped wave equations on partially rectangular domains. Math. Res. Lett. **14**(1), 35–47 (2007)
6. Burq, N., Joly, R.: Exponential decay for the damped wave equation in unbounded domains. Commun. Contemp. Math. **18**(06), 1650012 (2016)

7. Gearhart, L.: Spectral theory for contraction semigroups on hilbert space. Trans. Am. Math. Soc. **236**, 385–394 (1978)
8. Gesztesy, F., Jones, C.K.R.T., Latushkin, Y., Stanislavova, M.: A spectral mapping theorem and invariant manifolds for nonlinear Schrödinger equations. Indiana Univ. Math. J. **49**(1), 221–243 (2000). https://doi.org/10.1512/iumj.2000.49.1838
9. Huang, F.: Characteristic conditions for exponential stability of linear dynamical systems in hilbert spaces. Ann. Differ. Equ. **1**(1), 43–56 (1985)
10. Latushkin, Y., Shvydkoy, R.: Hyperbolicity of semigroups and Fourier multipliers. In: Borichev, A.A., Nikolski, N.K. (eds.) Systems, Approximation, Singular Integral Operators, and Related Topics. Operator Theory: Advances and Applications, vol. 129, pp. 341–363. Birkhäuser, Basel (2001)
11. Pazy, A.: Semigroups of linear operators and applications to partial differential equations, applied mathematical sciences, vol. 44. Springer-Verlag, New York (1983)
12. Phung, K.: Polynomial decay rate for the dissipative wave equation. J. Differ. Equ. **240**(1), 92–124 (2007)
13. Prüss, J.: On the spectrum of c_0-semigroups. Trans. Am. Amer. Math. Soc. **284**(2), 847–857 (1984)
14. Rauch, J., Taylor, M., Phillips, R.: Exponential decay of solutions to hyperbolic equations in bounded domains. Indiana Univ. Math. J. **24**(1), 79–86 (1974)
15. Stahn, R.: Optimal decay rate for the wave equation on a square with constant damping on a strip. Zeitschrift für angewandte Mathematik und Physik **68**(2), 36 (2017)
16. Wunsch, Jared: Periodic damping gives polynomial energy decay. Math. Res. Lett. **24**, 519–528 (2017)