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NLS and KdV Hamiltonian linearized operators: A priori bounds on the spectrum and optimal L^2 estimates for the semigroups

Harrison Gaebler, Milena Stanislavova *,1

Department of Mathematics, University of Kansas, 1460 Jayhawk Boulevard, Lawrence KS 66045–7523, United States of America

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ABSTRACT

Motivated by the NLS and KdV linearizations near traveling waves, we study general forms of such operators. We prove *a priori* bounds on the unstable spectrum, by showing that if any unstable spectrum exists, it is contained in a strip around the real axis, with an explicit estimate of its width in terms of the potentials. To the best of our knowledge, this is the first result of this nature in the literature. We show that all sufficiently large (relative to the potential) pure imaginary eigenvalues are necessarily simple. In the case of spectral stability, we show optimal, at most polynomial in time, L^2 bounds for the associated semigroups generated such linearized operators. As it is for finite matrices, the power rate matches the maximal size of any Jordan block minus one.

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1. Introduction

The study of spectral and orbital stability of traveling waves, both spatially periodic and solitary, of Hamiltonian PDEs has seen tremendous advances in the last three decades. Around 1990, Grillakis, Shatah and Strauss [1,2] described the stability theory of infinite dimensional Hamiltonian systems in an abstract formulation that is suitable for use in various different models of solitary waves in the presence of symmetry. See also [3], which is relevant to the equations discussed here. The corresponding generalized eigenvalue problem, which determines the spectral stability of such nonlinear waves, was studied later in [4] and [5], where "index counting" results about the number of eigenvalues with positive, negative and zero real parts were introduced. These were improved and generalized later in [6–9] and [10]. One can find additional important applications referenced in the excellent review by Kapitula and Deconinck [11]. At the same time there has been a myriad of novel results on the orbital stability of spatially periodic solutions for Hamiltonian PDEs. Some of these fall under the "energy methods" strategy, classically proposed by [1,2] and further developed for periodic waves in [12]. Examples of such are given by [13,14] for the NLS as well as [15–17] and many others. However, if one considers "subharmonic" perturbations, then the second variation of the energy functional contains additional negative eigenvalues and direct application of the standard energy method is impossible. In such situations, extra care and the use of inverse scattering techniques and higher level conserved quantities is needed in order to perform stability analysis of the waves. Examples of such results are [18] for NLS and [19] for Dirac solitons, see also [11] and [20]. These are not the main focus of our investigations here, although the methods we use have their origins in the same tradition and are of similar flavor. Rather, we want to investigate the NLS and the generalized KdV waves searching for optimal bounds for the associated semigroups in the cases when spectral stability holds.

The nonlinear Schrödinger equation is an ubiquitous model in quantum mechanics, which has been extensively studied in the literature. To fix notations, we consider it posed in the form

$$iu_t + \Delta u + f(|u|^2)u = 0, t > 0, x \in \mathbf{R}^d \text{ or } x \in [-L, L]^d. \quad (1)$$

for an appropriate non-linearity f. In both the unbounded and bounded cases, we assign the standard boundary conditions, expressed though the domain of the Laplacian $H^2(\mathbf{R}^d)$ ($H^2_{per}[-L, L]^d$ respectively), which makes the Laplacian a self-adjoint operator.

Another model that will be of interest is the generalized KdV equation, which we will only consider in the periodic setting. Namely, we seek real-valued solutions of the following PDE

$$u_t + u_{xxx} + \partial_x(f(u^2)u) = 0, x \in \mathbf{R} \text{ or } -L < x < L \quad (2)$$

Existence and uniqueness (and more generally well-posedness) for the Cauchy problems for (1) and (2), have been well-understood, after an extensive study in the last forty years, both in the infinite domains and in the periodic setting. For our purposes, it suffices to say that for nice enough functions f, local in time solutions exist, whenever the initial data is say in the class H^1. In many cases, (such as the focusing case, $f > 0 : f(z) =$

* Corresponding author.
E-mail addresses: hhg1@ku.edu (H. Gaebler), stanis@ku.edu (M. Stanislavova).

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$z^N + O(z^{N+1})$ for sufficiently large N), solutions may experience blow up in finite time, even for very smooth and well-localized initial data. On the other hand, in the defocussing cases and in the focusing cases with low power nonlinearity, H^1 conservation law, which is valid for both the NLS and the KdV models

$$E[u] = \int |\nabla u(x)|^2 dx - \int F(|u(x)|^2) dx, F' = f,$$

prevents blow-up in finite time. We will not discuss any further the well-posedness issues, as we focus our work on a different aspect of the dynamics, namely the behavior close to special solutions. Next, we describe some relevant background material.

1.1. Special solutions and the corresponding linearized problems

If one considers a standing wave solution, in the form $e^{i\omega t}\varphi(x)$ of the NLS model (1), with real-valued φ, we obtain the profile equation

$$-\Delta\varphi + \omega\varphi - f(\varphi^2)\varphi = 0,\ x \in \mathbf{R}^d \text{or } x \in [-L, L]^d. \tag{3}$$

Similarly, we find traveling wave solutions of the generalized KdV problem (2) as follows. We take the ansatz in the form $\varphi(x - \omega t)$. In the whole line case, we work under the assumption that φ vanishes at $\pm\infty$, while we assume periodic boundary conditions in the case $-L < x < L$. In each case, we integrate the associated ODE once and we obtain

$$-\varphi'' + \omega\varphi - f(\varphi^2)\varphi = a, x \in \mathbf{R} \text{ or } -L < x < L, \tag{4}$$

where $a = 0$ in the case $x \in \mathbf{R}$ and it is an arbitrary constant of integration otherwise. The elliptic problems (3) and (4) are well-known instances of the Newton's equation, which at least in one spatial dimension admits solutions in quadratures. We will henceforth assume that such solutions φ (with appropriate properties) exist, and we shall concentrate instead on the question of the dynamics of the data near them.

More specifically, we consider the linearization about these special solutions. Linearize around the standing wave $e^{i\omega t}\varphi$, that is take $u = e^{i\omega t}[\varphi + v]$ and further split the real and imaginary parts. Ignoring $O(|v|^2)$ terms, we obtain, after some algebraic manipulations, the linear system for $v_1 := \Re v$, $v_2 := \Im v$,

$$\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{5}$$

where

$$\mathcal{L}_1 = -\Delta + \omega - f(\varphi^2) - 2\varphi^2 f'(\varphi^2),$$
$$\mathcal{L}_2 = -\Delta + \omega - f(\varphi^2)$$

Passing to the eigenvalue form of the problem (5), $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{6}$$

For the generalized KdV problem, we take $u(t, x) = \varphi(x - \omega t) + v(t, x - \omega t)$, which after ignoring $O(v^2)$ terms, brings about the eigenvalue problem

$$v_t = \partial_x(-\partial_x^2 v + \omega v - (f(\varphi^2) + 2\varphi^2 f'(\varphi^2)v)) = \partial_x \mathcal{L}_1 v,$$

in the previous notations. Passing to the eigenvalue formulation $v \to e^{\lambda t} v$, yields

$$\partial_x \mathcal{L}_1 v = \lambda v \tag{7}$$

1.2. Motivation

As we saw above, both eigenvalue problems (6) and (7) are in the form $\mathcal{J}\mathcal{L}v = \lambda v$, where $\mathcal{J}^* = -\mathcal{J}$, $\mathcal{L}^* = \mathcal{L}$. More precisely, in the NLS case, we have

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix} \tag{8}$$

where $D(\mathcal{L}) = H^2 \times H^2$, while in the KdV example

$$\mathcal{J} = \partial_x, \mathcal{L} = \mathcal{L}_1, \tag{9}$$

with $D(\mathcal{L}_1) = H^2$. Note that in both examples, the operator $\mathcal{L}$ is a (matrix) Schrödinger operator in the form $\mathcal{L}_{1,2} = -\Delta + \omega - V_{1,2}$, where $V_{1,2}$ are generally smooth potentials. It is not hard to establish (and generally well-known), that the Hamiltonian linearized operators $\mathcal{J}\mathcal{L}$ of the form arising in (6) and (7) generate a C_0 semigroup on L^2, under very general conditions on the potentials.

It is of interest whether or not such semigroups satisfy the spectral mapping theorem. In particular, if the linearized operator $\mathcal{J}\mathcal{L}$ does not have unstable spectrum, is it true that the semigroup maps L^2 into itself, uniformly in time?

Another interesting question, which comes up often in applications[2] is where is the unstable spectrum of $\mathcal{J}\mathcal{L}$ possibly located? That is, is there a way to reduce the search for unstable spectrum to some *a priori* determined, possibly small region. More precisely, we pose and eventually address the following.

Question 1. *Under natural assumptions on the potentials V_1, V_2, give reasonable bounds on the location of the spectrum of $\mathcal{J}\mathcal{L}$ for the cases of the NLS semigroup (8) and the KdV semigroup (9).*

In fact, we consider this question in its more general form, namely, we consider general second order Schrödinger operators

$$\mathcal{L}_1 = -\Delta - V_1,$$
$$\mathcal{L}_2 = -\Delta - V_2.$$

and we are interested in the location of the spectrum as posed in Question 1. More specifically, assume that the semigroup generator $\mathcal{J}\mathcal{L}$ is either in the form

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathcal{L} = \begin{pmatrix} -\Delta - V_1 & 0 \\ 0 & -\Delta - V_2 \end{pmatrix},$$

or

$$\mathcal{J} = \partial_x, \mathcal{L} = -\partial_x^2 - V,$$

where V_1, V_2, V will be only assumed to be bounded functions (in the respective function classes), without any relation to the particular form as they arise in the linearized operators, (5) and (7). For this general class of operators, we also address the question for polynomial bounds. That is,

Question 2. *Assume spectral stability, that is $\sigma(\mathcal{J}\mathcal{L}) \subset i\mathbf{R}$. Under what extra assumptions on V_1, V_2 (V respectively), one has at most polynomial bounds. More precisely, does there exist a constant C, so that*

$$\|e^{t\mathcal{J}\mathcal{L}}\|_{L^2 \to L^2} < Ct^N? \tag{10}$$

1.3. Main results

We start with the Schrödinger case.

[2] For example in numerical runs for finding instabilities of Hamiltonian systems of this sort.

1.3.1. Schrödinger case

Our first result addresses Question 1, in the context of the Schrödinger semigroup (8).

Theorem 1 (*The Spectrum of the Linearized NLS Lies Inside Horizontal Strip*). *Let $d \geq 1, L > 0$ and $\Omega = \mathbf{R}^d$ or $\Omega = [-L, L]^d$. Assume that V_1, V_2 be real valued and bounded functions and set*

$$V := \frac{V_1 + V_2}{2}.$$

Then, the operator

$$\mathcal{J}\mathcal{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\Delta + V_1 & 0 \\ 0 & -\Delta + V_2 \end{pmatrix} \tag{11}$$

with $D(\mathcal{J}\mathcal{L}) = H^2(\Omega) \times H^2(\Omega)$ generates a C_0-semigroup on $L^2(\Omega)$.

More importantly, its spectrum satisfies

$$\sigma(\mathcal{J}\mathcal{L}) \subset i\mathbf{R} \cup \{z : \Re z \neq 0, |\Im z| \leq 2\|V\|_{L^\infty}\}.$$

Equivalently, all potential spectral instabilities of $\mathcal{J}\mathcal{L}$ lie inside the strip

$$\{z : |\Im z| \leq 2\|V\|_{L^\infty}\}.$$

Remark. Note that Theorem 1 applies to both the unbounded case $\Omega = \mathbf{R}^d$ and the periodic case $\Omega = [-L, L]^d$.

Next, we address the question for the time behavior of the semigroup's L^2 bounds, but only for the case $d = 1$ and periodic boundary conditions $\Omega = [-L, L]$. By rescaling, we can easily reduce the general case to $L = \pi$ or $\Omega = [-\pi, \pi]$, so we assume this henceforth.

Before we state the result, let us introduce some notations. Assume spectral stability for the operator $\mathcal{J}\mathcal{L}$, that is $\sigma(\mathcal{J}\mathcal{L}) \subset i\mathbf{R}$. By classical arguments, it is clear that $\sigma(\mathcal{J}\mathcal{L})$ consists of point spectrum of finite multiplicity only. Due to Hamiltonian invariance, for every $i\mu \in \sigma(\mathcal{J}\mathcal{L}) \cap i\mathbf{R}$, we have that $-i\mu \in \sigma(\mathcal{J}\mathcal{L}) \cap i\mathbf{R}$. In addition, each pair $\pm i\mu_j$ has some algebraic multiplicity n_j and a geometric multiplicity associated to it, which is the number of linearly independent eigenvectors $l_j \leq n_j$. In the case $l_j < n_j$, we have two copies of Jordan blocks to each pair, so let us denote their lengths by $n_j^1 \leq \cdots \leq n_j^{l_j}$, so that $n_j = n_1^1 + \cdots + n_j^{l_j}$. We have the following result.

Theorem 2. *Let $V_1, V_2; [-\pi, \pi] \to \mathbf{R}$ be bounded, real-valued functions and set $V = \frac{V_1+V_2}{2}$. Then the spectrum of $\sigma(\mathcal{J}\mathcal{L})$ consists of eigenvalues with finite multiplicity, accumulating only at infinity. Next, assume spectral stability, that is $\sigma(\mathcal{J}\mathcal{L}) \subset i\mathbf{R}$. Then all eigenvalues $\pm i\mu \in \sigma(\mathcal{J}\mathcal{L}) \cap i\mathbf{R}$, with $|\mu| > 2\max(\|V\|_{L^\infty}, 1)$ are simple. Denote the remaining, (finitely many) eigenvalues by $\{\pm i\mu_1, \ldots \pm i\mu_N\} = \sigma(\mathcal{J}\mathcal{L}) \cap \{i\mu : |\mu| \leq 2\max(\|V\|_{L^\infty}, 1)\}$. Then, in the notations above, there exists a constant C, so that*

$$\|e^{t\mathcal{J}\mathcal{L}} f\|_{L^2} \leq C t^{\max_{j\in[1,N]} n_j^{l_j} - 1} \|f\|_{L^2}. \tag{12}$$

In particular, if all eigenvalues of $\mathcal{J}\mathcal{L}$ are simple (or more generally their algebraic and geometric multiplicities match), then the semigroup is time uniformly bounded on L^2,

$$\sup_{0\leq t<\infty} \|e^{t\mathcal{J}\mathcal{L}}\|_{B(L^2)} \leq C.$$

Remarks.

- In the case of a general $L > 0$, the cutoff above which one finds only simple pure imaginary eigenvalues $\pm i\mu$, becomes $|\mu| > 2\max(\|V\|_{L^\infty}, \frac{L^2}{\pi^2})$.
- In order to state the power bound (12) in its current form, we need to impose the assumption of spectral stability — $\sigma(\mathcal{J}\mathcal{L}) \subset i\mathbf{R}$. While it is possible to state the result in the general case,[3] we chose this formulation, due to the fact that the main interest in (12) is in the case of spectral stability.
- The results in [10] offer similar uniform bound on the central manifold associated to (a much more general form) $\mathcal{J}\mathcal{L}$ — see Theorem 2.2, [10]. One has to note though that an application of these results to operators in the form (11), imply that the semigroup $e^{t\mathcal{J}\mathcal{L}}$, obeys the bound $\sup_{t>0} \|e^{t\mathcal{J}\mathcal{L}}\|_{H^1} \leq C t^{N-1} \|f\|_{H^1}$, where N is the size of the largest Jordan block associated to any purely imaginary eigenvalue of. On the other hand, our results apply to the more natural space L^2.
  More specifically, Lin and Zeng, [10] work within the Pontryagin framework, which necessitates that they use the norms induced by the domains of the quadratic forms of the self-adjoint piece $\mathcal{L}$, in this case H^1. Our approach is of completely different nature, as it permits the use of L^2 space. It is worth noting that the H^1 bounds in Theorem 2.2, [10] follow from (12).
- Lastly, the direct method for the proof of the power bound (12) should allow for other problems to be considered, which are not necessarily covered by [10]. Indeed, in the framework of Pontryagin spaces, one needs the self-adjoint operators to be semi-bounded. Thus, operators with infinite Morse index or with sign indefinite Hamiltonians, who will not be treatable with the Pontryagin's techniques, certainly can be analyzed with the methods developed in this paper.

1.3.2. KdV case

We now state the main results for the KdV case. As one can see, the results here — both the ones concerning Questions 1 and 2 are not as precise as those for the Schrödinger case. This is partly due to the fact that the operator $\mathcal{J} = \partial_x$ (and its inverse, on the space $L^2_0 = L^2 \cap \{u : \int u(x)dx = 0\}$) is less explicit to work with.

Theorem 3 (*The Spectrum of the Linearized KdV Lies Inside a Horizontal Strip*). *Let $V : [-\pi, \pi] \to \mathbf{R}$ be a real-valued potential, $V \in C^1[-\pi, \pi]$. Then, the spectrum of the operator $\partial_x(-\partial_x^2 - V)$ consists of eigenvalues with finite multiplicity, with only accumulation point at infinity. The operator generates a C_0 semigroup on $L^2[-\pi, \pi]$. In addition, there exists an absolute constant C, so that*

$$\sigma(\partial_x \mathcal{L}) \subset i\mathbf{R} \cup \{z : \Re z \neq 0, |\Im z| \leq C \max(\|V\|^3_{L^\infty}, 1)\}.$$

Equivalently, all instabilities of $\partial_x \mathcal{L}$ lie inside the strip $\{|\Im z| \leq C \max(\|V\|^3_{L^\infty}, 1)\}$.

Our next theorem shows at most polynomial in time growth for $e^{t\partial_x(-\partial_x^2 - V)}$, assuming spectral stability.

Theorem 4. *Let V be a real-valued potential. Assume spectral stability, that is*

$\sigma(\partial_x(-\partial_x^2 - V)) \subset i\mathbf{R}$. Then,

$$\|e^{t\partial_x(-\partial_x^2 - V)} f\|_{L^2} \leq C t^{N-1} \|f\|_{L^2}, \tag{13}$$

where N is the size of the largest Jordan block associated to any purely imaginary eigenvalue of $\partial_x(-\partial_x^2 - V)$.

If all (purely imaginary) eigenvalues of $\mathcal{J}\mathcal{L}$ are simple or more generally, their algebraic and geometric multiplicities coincide, then the semigroup is time-uniformly bounded on L^2,

$$\sup_{0\leq t<\infty} \|e^{t\mathcal{J}\mathcal{L}}\|_{B(L^2)} \leq C.$$

[3] With corresponding exponentially decaying or growing factors, equal to the maximal real part of the eigenvalues in $\sigma(\mathcal{J}\mathcal{L})$.

The plan of the paper is as follows. In Section 2 we collect some preliminary and well-known facts about the operators we are working with, their specta and long-time behavior. Importantly, we state and prove Gomilko type bounds for self-adjoint operators, an interesting result that will be very useful for the rest of the paper. In Section 3 we construct resolvents for the NLS operator, while in Section 4 we do the same for the KdV problem. In Section 5 we use Gomilko's criteria to prove uniform bounds for the NLS semigroup by splitting the cases of low frequencies from those of high frequency. Finally, Section 6 does the same for the KdV bounds.

2. Preliminaries

We start with some basics regarding Fourier series. For a locally integrable function $f : [-L, L] \to \mathbb{C}$, define its Fourier coefficients

$$\hat{f}(k) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi\frac{x}{L}}dx,\ k = 0, \pm1, \pm2, \ldots.$$

Then, in $L^2[-L, L]$ sense,

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ik\pi\frac{x}{L}}, \quad -L \le x \le L.$$

and the Plancherel's identity takes the form $\|f\|_{L^2}^2 = 2L\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2$. For every integer k, define the symmetric operators

$$P_k f(x) = \hat{f}(k)e^{ik\pi\frac{x}{T}},\ P_{|k|} = P_k + P_{-k},\ P_{\neq|k|} = Id - P_{|k|}.$$

It is well-known fact that for a self-adjoint (generally unbounded) operator $\mathcal{H}$, acting on a Hilbert space H, there is the resolvent bound

$$\|(\mathcal{H} - (\mu + iq))^{-1}\|_{B(H)} = \frac{1}{dist(\mu + iq, \sigma(\mathcal{H}))} \le |q|^{-1}. \tag{14}$$

for all $\mu, q \in \mathbf{R}$.

2.1. Some semigroups basics

We work with the standard notion of strongly continuous or C_0 semigroup, namely that on a fixed Banach[4] space X, there is a family of bounded operators $\{T(t)\}_{t\ge0} \subset B(X)$, so that $T(0) = Id$, $T(t+s) = T(t)T(s)$ and for every $x \in X : \lim_{t\to0+} \|T(t)x-x\|_X = 0$. It is well-known that such semigroups are associated to (generally unbounded) linear operators, $\mathcal{A}$ called generators, defined via

$$D(\mathcal{A}) = \{x \in X : \lim_{t\to0+} \frac{T(t)x - x}{t} - \text{ exists }\},\ \mathcal{A}x := \lim_{t\to0+} \frac{T(t)x - x}{t},$$

so that $T(t)f = e^{t\mathcal{A}}f$, in appropriate sense, namely as the unique solution, at time t of the initial value problem $u_t = \mathcal{A}u$, $u(x, 0) = f(x)$. Such operators $\mathcal{A}$ must obey a number of properties to do so, but we will not touch upon that. Importantly, it is well-known that for each C_0 semigroup there is an estimate $\|e^{t\mathcal{A}}\|_{B(X)} \le Ce^{\omega t}$, which allows one to introduce the growth bound

$$\omega_0(\mathcal{A}) = \inf\{\omega : \|e^{t\mathcal{A}}\|_{B(X)} \le Ce^{\omega t}\}.$$

If for every $\delta > 0$, there is C_δ, so that the bound $\|e^{t\mathcal{A}}\|_{B(X)} \le C_\delta e^{(\gamma+\delta)t}$ holds, then

$$\omega_0(\mathcal{A}) \le \gamma.$$

An easy to verify condition for C_0 semigroup generation is dissipativity, which we restrict to Hilbert spaces.

[4] Which usually for us will be a Hilbert space.

Theorem 5. *Let H be a Hilbert space and $(\mathcal{A}, D(\mathcal{A}))$ is closed, densely defined operator. Then, the following are equivalent:*

(1) *$\mathcal{A}$ is a dissipative operator. That is*

$$\Re\langle \mathcal{A}x, x\rangle \le 0. \tag{15}$$

(2) *$\mathcal{A}$ generates a semigroup of contractions.*

The theorem is essentially a corollary of the Lumer–Phillips theorem. The necessity is well-documented, see for example Proposition 3.23, p. 88, [21]. The sufficiency of the condition is as follows: (15) implies that $\mathcal{A}^*$ is dissipative as well. Then, one can show that for all $\lambda > 0$, $\lambda - \mathcal{A}$ is surjective, hence by Lumer–Phillips one gets that $\mathcal{A}$ generates a semi-group of contractions. This argument is carried out in full detail in Corollary 3.17, p. 84, [21].

An easy corollary of Theorem 5 is the following.

Corollary 1. *Let H be a Hilbert space and the closed, densely defined operator $(\mathcal{B}, D(\mathcal{B}))$ satisfies*

$$\Re\langle \mathcal{B}x, x\rangle \le \omega\|x\|^2. \tag{16}$$

for some $\omega \in \mathbf{R}$ and every $x \in D(\mathcal{B})$. Then $\mathcal{B}$ generates a C_0 semigroup, with growth bound $\omega_0(\mathcal{B}) \le \omega$.

Corollary 1 follows by applying Theorem 5 to $\mathcal{A} := \mathcal{B} - \omega$.

2.2. The $\mathcal{JL}$ operators generate C_0 semigroups

In both cases of NLS and KdV problems, the operators are trivially relatively bounded perturbations of the corresponding constant coefficient operators. We now prove the claims regarding semi-group generation in Theorems 1 and 3.

Let us first discuss the NLS case, that is the semi-group introduced in (11), when V_1, V_2 are bounded potentials.

2.2.1. The NLS case

Our main tool in this will be the bounded perturbation theorem, see Theorem 1.3, p. 158, [21], which states that $(\mathcal{A}, D(\mathcal{A}))$ generates a semi-group and $\mathcal{B}$ is bounded, then $(\mathcal{A} + \mathcal{B}, D(\mathcal{A}))$ generates semi-group as well. As we apply this result to the operators

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix},\ \mathcal{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$$

and $\mathcal{B}$ is bounded, it clearly suffices to show that $(\mathcal{A}, H^2(\Omega) \times H^2(\Omega))$ generates a semi-group on $L^2(\Omega) \times L^2(\Omega)$. Elementary calculations show

$$e^{t\mathcal{A}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin(-t\Delta) + Id_2\cos(-t\Delta),$$

where the operators $\sin(-t\Delta), \cos(-t\Delta)$ are defined via the semi-groups $e^{\pm it\Delta}$, which in turn can be realized via the Fourier inversion/Fourier series formulas. Thus, $\mathcal{A}$ generates a C_0 semi-group and hence the $\mathcal{JL}$ operator in the form (11) generates one as well.

2.2.2. The KdV case

In the KdV case, we need to rely on Corollary 1 instead. We have $\mathcal{A} = \partial_x(-\partial_x^2+V)$, a closed, densely defined operator, $D(\mathcal{A}) = H^3([-L, L])$, so we need to verify (16). We clearly have for every $f \in H^3(\Omega)$, $\Re\langle -\partial_x^3 f, f\rangle = 0$, so it remains to estimate

$$\Re\langle \partial_x(Vf), f\rangle = -\Re\int_\Omega Vf\bar{f}'dx = -\frac{1}{2}\int_\Omega V\partial_x|f|^2dx$$
$$= \frac{1}{2}\int_\Omega V'(x)|f(x)|^2dx \le \|V'\|_{L^\infty}\|f\|^2.$$

Thus, we have established (16), with $\omega = \|V'\|_{L^\infty}$ and $\partial_x(-\partial_x^2+V)$ generates a semigroup.

2.3. Gearhart–Prüss and Gomilko's theorems

We begin with a statement of the celebrated Gearhart–Prüss theorem — see Theorem 2.16, p. 97, [22].

Theorem 6. *Let $\mathcal{A}$ generate a strongly continuous semi-group on a complex Hilbert space H. Then, the following are equivalent*

- $\mathcal{C}_+ := \{\Re z > 0\} \subset \rho(\mathcal{A})$ *and*
$$\sup_{z:\Re z>0} \|(z-\mathcal{A})^{-1}\|_{B(H)} < \infty$$
- $\omega_0(\mathcal{A}) < 0$ *or equivalently, there exist $\delta > 0$ and a constant C, so that*
$$\|e^{t\mathcal{A}}\|_{B(H)} \leq Ce^{-\delta t}. \tag{17}$$

Let us make some comments regarding this formulation of the theorem, as there are various (essentially) equivalent versions available in the literature. To start, we can use a straightforward corollary of the inversion formula for the Laplace transform to represent $(\mathcal{A}-z)^{-1} = -\int_0^\infty e^{-zt}e^{t\mathcal{A}}dt$, whenever $\mathcal{A}$ satisfies (17) and $z : \Re z > -\delta$, which means that in this case $\{\Re z > -\delta\} \subset \rho(\mathcal{A})$. In particular, $\{\Re z = 0\} \subset \rho(\mathcal{A})$. Conversely, one usually checks the following sufficient condition for (17), namely $\{\Re z \geq 0\} \subset \rho(\mathcal{A})$ and $\sup_{\mu\in\mathbf{R}} \|(i\mu-\mathcal{A})^{-1}\|_{B(H)} < \infty$. In fact, we have the following corollary.

Corollary 2 (*Gearhart–Prüss - Second Version*)**.** *Let $\mathcal{A}$ generate a strongly continuous semi-group on a complex Hilbert space H. Assume that $\{\Re z \geq 0\} \subset \rho(A)$ and*

$$M := \sup_{\mu\in\mathbf{R}} \|(i\mu-\mathcal{A})^{-1}\|_{B(H)} < \infty.$$

Then, $\omega_0(\mathcal{A}) < 0$.

Remark. As mentioned above, the converse is also true, since $\omega_0(\mathcal{A})$ implies $\{\Re z > -\delta\} \subset \rho(\mathcal{A})$ and then, one gets the uniform resolvent bounds on the imaginary axes by the representation $(\mathcal{A}-z)^{-1} = -\int_0^\infty e^{-zt}e^{t\mathcal{A}}dt$.

Proof. First, by the resolvent identity we have that for all $\delta \in \mathbf{R}$,

$$(\delta+i\mu-\mathcal{A})^{-1}(1+\delta(i\mu-\mathcal{A})^{-1}) = (i\mu-\mathcal{A})^{-1}.$$

By Neumann series expansions, for all $\delta : |\delta| < \delta_0 := \frac{1}{2M}$, we have that $(1+\delta(i\mu-\mathcal{A})^{-1})$ is invertible for all $\mu \in \mathbf{R}$ and $\|(1+\delta(i\mu-\mathcal{A})^{-1})^{-1}\|_{B(H)} \leq 2$. Thus, $(\delta+i\mu-\mathcal{A})^{-1} = (i\mu-\mathcal{A})^{-1}(1+\delta(i\mu-\mathcal{A})^{-1})^{-1}$ and

$$\sup_{\mu\in\mathbf{R}} \|(\delta+i\mu-\mathcal{A})^{-1}\|_{B(H)} \leq 2\sup_{\mu\in\mathbf{R}} \|(i\mu-\mathcal{A})^{-1}\|_{B(H)} = 2M.$$

Thus, we have shown

$$\sup_{z:-\delta_0\leq\Re z\leq\delta_0} \|(z-\mathcal{A})^{-1}\|_{B(H)} < \infty. \tag{18}$$

Thus, $s_0(\mathcal{A}) \leq -\delta_0 < 0$. Thus, one has the Laplace transform representation

$$\begin{aligned} T(t)x &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(a+i\mu)t}(a+i\mu-\mathcal{A})^{-1}d\mu \\ &= \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{(a+i\mu)t}(a+i\mu-\mathcal{A})^{-2}d\mu \end{aligned} \tag{19}$$

for all $a > s_0(\mathcal{A})$, see Theorem 3.8, [22].

For the rest, we follow the proof of the Gearhart–Prüss theorem from [22], see Theorem 2.16, p. 97, with minor modifications. We need to show

$$\int_{-\infty}^{\infty} |\langle(is-\mathcal{A})^{-2}x, y\rangle|ds < \infty \tag{20}$$

for all $x, y \in H$, see Theorem 2.15, [22].

For $a > \omega_0(\mathcal{A})$, due to the representation of the resolvent $(a+i\mu-\mathcal{A})^{-1}$ as a Laplace transform of $e^{-at}T(t)$, (19), we have from Plancherel's identity

$$\int_{-\infty}^{\infty} \|(a+i\mu-\mathcal{A})^{-1}x\|^2 d\mu = \int_0^{\infty} e^{-2at}\|T(t)x\|^2 dt < \infty. \tag{21}$$

for each $x \in H$. By the resolvent identity

$$\|(i\mu-\mathcal{A})^{-1}\| = \|(I+a(i\mu-\mathcal{A})^{-1})(a+i\mu-\mathcal{A})^{-1}\| \leq (1+Ma)\|(a+i\mu-\mathcal{A})^{-1}\|$$

whence from (21), for all $x \in H$,

$$\int_{-\infty}^{\infty} \|(i\mu-\mathcal{A})^{-1}x\|^2 d\mu < C\|x\|^2 \tag{22}$$

Applying the same arguments to A^*, we obtain, for all $x \in H$,

$$\int_{-\infty}^{\infty} \|(i\mu-\mathcal{A}^*)^{-1}x\|^2 d\mu < C\|x\|^2 \tag{23}$$

Clearly, (23) and (22) imply (20), hence by (19)

$$|\langle T(t)x, y\rangle| \leq \frac{1}{2\pi t}\int_{-\infty}^{\infty} |\langle(i\mu-\mathcal{A})^{-2}x, y\rangle|d\mu \leq \frac{C\|x\|\|y\|}{t}$$

Hence $\|T(t)\| \to 0$ and this implies $\omega_0(\mathcal{A}) < 0$. □

As useful as this Gearhart–Pruss result is, it fails to distinguish between exponentially decaying semigroups (i.e. growth bound $\omega(\mathcal{A}) < 0$) and uniformly in time bounded semigroups, which means slightly more than $\omega_0(\mathcal{A}) = 0$. Such result is available in the literature. We state a precise version of it, due to Gomilko, [23], see also Theorem 1.1, p. 82, [22].

Theorem 7 (*Gomilko*)**.** *Let $\mathcal{A}$ generate a C_0 semigroup on a Hilbert space H. Then, the following are equivalent:*

- $\mathcal{C}_+ \subset \rho(\mathcal{A})$ *and there is a constant C, so that for any $f \in H$*
$$\sup_{\delta>0} \delta\int_{-\infty}^{\infty} [\|(\mathcal{A}-(\delta+i\mu))^{-1}f\|_H^2 + \|(\mathcal{A}^*-(\delta+i\mu))^{-1}f\|_H^2]d\mu \leq C\|f\|_H^2. \tag{24}$$
- $e^{t\mathcal{A}}$ *is uniformly bounded on H, i.e.*
$$\sup_{0<t<\infty} \|e^{t\mathcal{A}}\|_{H\to H} < \infty.$$

Note that it suffices to assume the condition (24) only for all small enough $\delta > 0$, say $0 < \delta < 1$. That is, if $\mathcal{C}_+ \subset \rho(\mathcal{A})$ and

$$\sup_{0<\delta<1} \delta\int_{-\infty}^{\infty} [\|(\mathcal{A}-(\delta+i\mu))^{-1}f\|_H^2 + \|(\mathcal{A}^*-(\delta+i\mu))^{-1}f\|_H^2]d\mu \leq C\|f\|_H^2. \tag{25}$$

then $e^{t\mathcal{A}}$ is uniformly bounded.

Besides obvious applications to uniformly bounded semigroups, Gomilko's criteria may be used to identify equality in the growth bounds as follows. Suppose that a condition reminiscent of (24)

$$\begin{aligned} &\sup_{\delta>\delta_0} \int_{-\infty}^{\infty} [\|(\mathcal{A}-(\delta+i\mu))^{-1}f\|_H^2 + \|(\mathcal{A}^*-(\delta+i\mu))^{-1}f\|_H^2]d\mu \\ &\leq C_{\delta_0}\|f\|_H^2, \end{aligned} \tag{26}$$

holds for every $\delta_0 > 0$. Then, we claim that for all $\epsilon > 0$, there is C_ϵ, so that $\|e^{t\mathcal{A}}\|_{B(H)} \le C_\epsilon e^{\epsilon t}$. Indeed, (26) implies

$$\sup_{0<\alpha<1} \alpha \int_{-\infty}^{\infty} [\|(\mathcal{A} - \delta_0 - (\alpha + i\mu))^{-1} f\|_H^2 + \|(\mathcal{A}^* - \delta_0 - (\alpha + i\mu))^{-1} f\|_H^2] d\mu \le C_{\delta_0} \|f\|_H^2,$$

By (25), this means that $\mathcal{A} - \delta_0$ generates uniformly bounded semigroup for every $\delta_0 > 0$. That is, $\|e^{t\mathcal{A}}\|_{B(H)} \le C_{\delta_0} e^{\delta_0 t}$. Thus, according to our previous remarks, we must have that $\omega_0(\mathcal{A}) \le 0$.

2.4. Gomilko type bounds for self-adjoint operators

The following lemma is a consequence of Theorem 7, but we present its direct proof in order to introduce some useful techniques for our subsequent arguments.

Lemma 1. *Let $\mathcal{M}$ be a self-adjoint operator on a Hilbert space H. Then,*

$$\sup_{\delta>0} \delta \int_{-\infty}^{\infty} \|(\mathcal{M} - \mu + i\delta)^{-1} f\|_H^2 d\mu \le C\|f\|_H^2.$$

Proof. The lemma follows by applying abstract results as follows. By the Stone's theorem, $i\mathcal{M}$ generates a group of isometries, $\|e^{it\mathcal{M}}\|_{B(H)} = 1$. Thus, by the necessity in Gomilko's theorem

$$\sup_{\delta>0} \delta \int_{-\infty}^{\infty} \|(i\mathcal{M} - (\delta + i\mu))^{-1} f\|_H^2 d\mu \le C\|f\|_H^2.$$

This is of course equivalent to the claim in the Lemma.

We proceed with a simple direct proof, to illustrate some ideas that will be useful later on. Consider the spectral decomposition for $\mathcal{M}$, namely $\mathcal{M}f = \int_{\sigma(\mathcal{M})} \lambda dE_\lambda$. We have

$$\int_{-\infty}^{\infty} \|(\mathcal{M} - \mu + i\delta)^{-1} f\|_H^2 d\mu$$
$$= \int_{-\infty}^{\infty} \langle (\mathcal{M} - \mu + i\delta)^{-1} f, (\mathcal{M} - \mu + i\delta)^{-1} f \rangle d\mu =$$
$$= \int_{-\infty}^{\infty} \langle (\mathcal{M} - \mu + i\delta)^{-1} (\mathcal{M} - \mu - i\delta)^{-1} f, f \rangle d\mu$$
$$= \int_{-\infty}^{\infty} \int_{\sigma(\mathcal{M})} \frac{1}{|\lambda - \mu|^2 + \delta^2} d\langle E_\lambda f, f \rangle d\mu$$
$$= \int_{\sigma(\mathcal{M})} \langle E_\lambda f, f \rangle \left(\int_{-\infty}^{\infty} \frac{1}{|\lambda - \mu|^2 + \delta^2} d\mu \right) d\lambda$$
$$= \pi \delta^{-1} \int_{\sigma(\mathcal{M})} \langle E_\lambda f, f \rangle d\lambda = \pi \delta^{-1} \|f\|_{L^2}^2. \quad \square$$

3. Construction of the NLS resolvent and absence of unstable spectrum outside a strip

We start with a derivation of a convenient representation of the resolvent of the operator $\mathcal{J}\mathcal{L}$ in the form (11).

3.1. Resolvent formulas for NLS

Clearly, since $\mathcal{J}^{-1} = -\mathcal{J}$, we have

$$(\mathcal{J}\mathcal{L} - (\delta + i\mu))^{-1} = (\mathcal{J}[\mathcal{L} + \mathcal{J}(\delta + i\mu)])^{-1} = -[\mathcal{L} + \mathcal{J}(\delta + i\mu)]^{-1}\mathcal{J},$$
$$((\mathcal{J}\mathcal{L})^* - (\delta + i\mu))^{-1} = ([\mathcal{L} - \mathcal{J}(\delta + i\mu)]\mathcal{J})^{-1} = -\mathcal{J}[\mathcal{L} - \mathcal{J}(\delta + i\mu)]^{-1}$$

Clearly, in order to study the resolvent operator $(\mathcal{J}\mathcal{L} - (\delta + i\mu))^{-1}$, it suffices to construct $(\mathcal{L} + \mathcal{J}(\delta + i\mu))^{-1}$. Write $z = [\mathcal{L} \pm \mathcal{J}(\delta + i\mu)]^{-1} f$, that is

$$[\begin{pmatrix} -\Delta - V_1 & 0 \\ 0 & -\Delta - V_2 \end{pmatrix} \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\delta + i\mu)] z = f. \quad (27)$$

Representing

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = S \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} S^{-1}, \quad S = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

and some algebraic manipulations leads to the equivalent to (27) form

$$\begin{pmatrix} -\Delta - V \pm (\mu - \delta i) & -V \\ V & -\Delta - V \mp (\mu - \delta i) \end{pmatrix} z = f \quad (28)$$

where $V := \frac{V_1 + V_2}{2}$. At this point, it is clear that the estimates required for the two cases are symmetric, so we consider the resolvent in the form

$$\begin{pmatrix} -\Delta - V - \mu + i\delta & -V \\ V & -\Delta - V + \mu - i\delta \end{pmatrix} z = f \quad (29)$$

Moreover, the case $\mu > 0$ and $\mu < 0$ are symmetric as well, so we just concentrate on the case $\mu > 0$. We have

$$\begin{cases} (-\Delta - V - \mu + i\delta) z_1 - V z_2 = f_1 \\ (-\Delta - V + \mu - i\delta) z_2 + V z_1 = f_2 \end{cases} \quad (30)$$

Clearly, for $\mu > \mu_0 \gg 1$, the second equation is easily resolvable, since $(-\Delta - V + \mu) > \mu - V > \frac{\mu}{2}$, for large enough μ_0. In fact, by elementary estimates for self-adjoint operators

$$\|(-\Delta - V + \mu)^{-1}\|_{L^2 \to L^2} + \|(-\Delta - V + \mu - i\delta)^{-1}\|_{L^2 \to L^2} \le C\mu^{-1},$$

for an absolute constant C. We can express z_2 from the second equation in (30) as follows

$$z_2 = (-\Delta - V + \mu - i\delta)^{-1} f_2 - (-\Delta - V + \mu - i\delta)^{-1} V z_1, \quad (31)$$

for as long as $(-\Delta - V + \mu - i\delta)^{-1}$ exists.

Using this relation in the equation for z_1 in (30), we obtain the following equation for z_1

$$(-\Delta - V - \mu + i\delta) z_1 + V(-\Delta - V + \mu - i\delta)^{-1} V z_1 = V(-\Delta - V + \mu - i\delta)^{-1} f_2 + f_1.$$

Note that this last equation is autonomous for z_1, which means that we can concentrate on it for the time being and then use the results in (31) to derive the equation for z_2.

By the resolvent identity

$$(-\Delta - V + \mu - i\delta)^{-1} = (-\Delta - V + \mu)^{-1} - i\delta(-\Delta - V + \mu - i\delta)^{-1}(-\Delta - V + \mu)^{-1},$$

whence we finally derive an equation for z_1 in the form,

$$(-\Delta - V + V(-\Delta - V + \mu)^{-1} V - \mu + i\delta) z_1 =$$
$$= i\delta V(-\Delta - V + \mu - i\delta)^{-1}(-\Delta - V + \mu)^{-1} V z_1$$
$$+ \quad V(-\Delta - V + \mu - i\delta)^{-1} f_2 + f_1.$$

This will be our final form of the resolvent equation, so for convenience we introduce the two self-adjoint operators

$$\mathcal{H}_0 := -\Delta - V$$
$$\mathcal{H} := -\Delta - V + V(-\Delta - V + \mu)^{-1} V$$

Note that the operator appearing in front of z_1, $(\mathcal{H} - \mu + i\delta)$ is invertible, as $\sigma(\mathcal{H})$ is real. Clearly, the norm of the inverse may be large, depending on μ (namely, if $dis(\mu, \sigma(\mathcal{H})) = O(\delta)$), namely of order δ^{-1}, which is the challenge in its estimation. On the other hand, note that the other operators in the formula in front of z_1 have norms $O(\mu^{-2})$, which allows one, at least for large μ to invert the operator on the left of (32) via von Neumann.

In any case, we can rewrite the equivalent system for z_1, z_2 as follows

$$(I - i\delta(\mathcal{H} - \mu + i\delta)^{-1}V(\mathcal{H}_0 + \mu - i\delta)^{-1}(\mathcal{H}_0 + \mu)^{-1}V)z_1(\mu) = (\mathcal{H} - \mu + i\delta)^{-1}V(\mathcal{H}_0 + \mu - i\delta)^{-1}f_2 + (\mathcal{H} - \mu + i\delta)^{-1}f_1. \quad (32)$$

$$z_2(\mu) = (\mathcal{H}_0 + \mu - i\delta)^{-1}f_2 - (\mathcal{H}_0 + \mu - i\delta)^{-1}Vz_1(\mu). \quad (33)$$

We have shown the following

Proposition 1. *Let $\delta > 0$. Then, the resolvent $(\mathcal{L} - \mathcal{J}(\delta + i\mu))^{-1}$ for $\mu > \|V\|_{L^\infty}$ is given by the implicit relations (32) and (33).*

3.2. Absence of spectrum for the NLS generator in $\{z : \Re\lambda \neq 0, |\Im\lambda| > 2\|V\|_{L^\infty}\}$

We are now ready to show that there cannot be unstable spectrum of the $\mathcal{J}\mathcal{L}$ with imaginary part larger than $2\|V\|_{L^\infty}$.

Proposition 2. *Let V_1, V_2 be bounded real-valued potentials, $V = V_1 + V_2$. Then,*

$$\sigma(\mathcal{J}\mathcal{L}) \cap \{\Re\lambda \neq 0, |\Im\lambda| > 2\|V\|_{L^\infty}\} = \emptyset.$$

In other words, if $\lambda \in \sigma(\mathcal{J}\mathcal{L})$, then either $\Re\lambda = 0$ or $|\Im\lambda| \leq 2\|V\|_{L^\infty}$.

Proof. The claim of the proposition will follow from the invertibility of $\mathcal{J}\mathcal{L} - (\delta + i\mu)$ for all $\delta \neq 0, |\mu| > 2\|V\|_{L^\infty}$. By Hamiltonian symmetries, it suffices to consider the case when $\delta > 0, \mu > 2\|V\|_{L^\infty}$.

As we have discussed above, the invertibility of $\mathcal{J}\mathcal{L} - (\delta + i\mu)$ is equivalent to bounds for z_1, z_2 in (32) and (33) in the form

$$\|z_1\|_{L^2} + \|z_2\|_{L^2} \leq C_{\mu,\delta}(\|f_1\|_{L^2} + \|f\|_{L^2}). \quad (34)$$

By the self-adjointness of $\mathcal{H}_0$ and the fact that $\mu - V \geq \frac{\mu}{2}$ (since $\mu > 2\|V\|_{L^\infty}$), so $-\Delta + \mu - V > \frac{\mu}{2}$, we have the bounds

$$\|(\mathcal{H}_0 + \mu)^{-1}\|_{B(L^2)} < \frac{2}{\mu}, \|(\mathcal{H}_0 + \mu - i\delta)^{-1}\|_{B(L^2)} < \frac{2}{\mu}.$$

Further, by the self-adjointness of $\mathcal{H}$, we have that

$$\|(\mathcal{H} - \mu + i\delta)^{-1}\| \leq \delta^{-1}.$$

All in all, we obtain the following estimate

$$\|\delta(\mathcal{H}-\mu+i\delta)^{-1}V(\mathcal{H}_0+\mu-i\delta)^{-1}(\mathcal{H}_0+\mu)^{-1}V\|_{B(L^2)} \leq \frac{4\|V\|_{L^\infty}^2}{\mu^2} < 1,$$

according to $\mu > 2\|V\|_{L^\infty}$. It follows that the operator on the left-hand side of (32), $I - i\delta(\mathcal{H}-\mu+i\delta)^{-1}V(\mathcal{H}_0+\mu-i\delta)^{-1}(\mathcal{H}_0+\mu)^{-1}V$ is invertible and

$$\|(I - i\delta(\mathcal{H} - \mu + i\delta)^{-1}V(\mathcal{H}_0 + \mu - i\delta)^{-1}(\mathcal{H}_0 + \mu)^{-1}V)^{-1}\|_{B(L^2)} \leq \frac{1}{1 - 4\mu^{-2}\|V\|_{L^\infty}^2}.$$

As a consequence, we take $\|\cdot\|_{L^2}$ in (32) and we obtain the bounds

$$\|z_1\|_{L^2} \leq \frac{2\mu^{-1}\delta^{-1}\|V\|_{L^\infty}\|f_2\|_{L^\infty} + \delta^{-1}\|f_1\|_{L^2}}{1 - 4\mu^{-2}\|V\|_{L^\infty}^2}. \quad (35)$$

Using this estimate in (33), we obtain the bound for z_2,

$$\|z_2\|_{L^2} \leq 2\mu^{-1}\|f_2\|_{L^2} + 2\mu^{-1}\|V\|_{L^\infty}\|z_1\|_{L^2} \leq$$
$$\leq 2\mu^{-1}\|f_2\|_{L^2} + 2\mu^{-1}\|V\|_{L^\infty}\frac{2\mu^{-1}\delta^{-1}\|V\|_{L^\infty}\|f_2\|_{L^\infty} + \delta^{-1}\|f_1\|_{L^2}}{1 - 4\mu^{-2}\|V\|_{L^\infty}^2}.$$

This shows absence of spectrum of $\sigma(\mathcal{J}\mathcal{L})$ off the imaginary axes (as $\delta \neq 0$), in the strip $\Im z > 2\|V\|_{L^\infty}$. □

Remark. Note that the formula (35) and the estimate for z_2 imply that for a fixed δ, there is uniform bound for $(\mathcal{J}\mathcal{L} - (\delta + i\mu))^{-1}$. More precisely, for each $\delta > 0$, there exists C_δ, so that

$$\sup_{|\mu| > \|V\|_{L^\infty}} \|(\mathcal{J}\mathcal{L} - (\delta + i\mu))^{-1}\| \leq C. \quad (36)$$

4. Resolvent construction for the KdV problem and absence of unstable spectrum outside a strip

We need to study the resolvent of the operator $\partial_x\mathcal{L}_1 = \partial_x(-\partial_x^2 - V)$ on a periodic interval $[-L, L]$, where V is a real-valued potential. By rescaling, we can reduce to the case $L = \pi$, so we assume this henceforth. Thus,

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}, \hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx,$$

Introduce the spectral parameter $\delta + i\mu$, where we take $\delta > 0, \mu > 0$, since the case $\mu < 0$ is similarly explored based on symmetry considerations.

We would like to construct the resolvent, whenever possible. Most importantly, we would like to see whether the operator $\partial_x(-\partial_x^2 - V) - (\delta + i\mu)$ has an inverse in $L^2_{per.}[-\pi, \pi]$ and if so, what is the dependence of $(\partial_x(-\partial_x^2 - V) - (\delta + i\mu))^{-1}$ on the parameters δ, μ especially as $\delta \to 0+$, while $\mu \gg 1$.

To set things concretely, let G be a given function and introduce, for each $\mu \in \mathbf{R}$, $F := [\partial_x(-\partial_x^2 - V) - (\delta + i\mu)]^{-1}G$. As we have shown, $\delta + i\mu \in \rho(\partial_x(-\partial_x^2 - V) - (\delta + i\mu))$, so F is well-defined. That is

$$[\partial_x(-\partial_x^2 - V) - (\delta + i\mu)]F = G, -\pi \leq x \leq \pi. \quad (37)$$

We need to show

$$\sup_{\delta:0<\delta<1} \delta\int_{-\infty}^{\infty} \|F(\mu)\|_{L^2}^2 d\mu \leq C\|G\|_{L^2}^2.$$

The first step is to integrate (37) in $[-\pi, \pi]$, which implies

$$\hat{F}(0) = -\frac{1}{\delta + i\mu}\hat{G}(0), \quad (38)$$

which resolves the zero modes completely, since $\int_{-\infty}^{\infty} \frac{1}{\mu^2+\delta^2}d\mu = \pi\delta^{-1}$.

From now, assume without loss of generality that both F, G are mean free, i.e. $\hat{F}(0) = \hat{G}(0) = 0$. We apply ∂_x^{-1} in (37), so we obtain

$$[-\partial_x^2 - V - i\mu\partial_x^{-1} - \delta\partial_x^{-1}]F = \partial_x^{-1}G, -\pi \leq x \leq \pi. \quad (39)$$

Note that the operator $-\partial_x^2 - V - i\mu\partial_x^{-1}$ is self-adjoint, while $\delta\partial_x^{-1}$ is skew-symmetric. To simplify the calculations, take $\mu := \nu^3$.

4.1. Some heuristics and strategy

Ignoring for a moment the potential term V and the non-self adjoint term $\delta\partial_x^{-1}$, we see that the linear term's dispersion is of the form

$$k^2 - \frac{\mu}{k} = \frac{1}{k}(k^3 - \nu^3) = \frac{1}{k}(k - \nu)(k^2 + k\nu + \nu^2).$$

Thus, the modulus of the dispersion function is small (and potentially zero), only k is very close to ν. In order to exploit this quantitatively, let $]\nu[$ denote the closest integer to the real number ν. Then, for $k \neq k_0(\nu)$, we have that $|k - \nu| \geq \frac{1}{2}$ and hence

$$\left|k^2 - \frac{\mu}{k}\right| \geq \frac{1}{4}\frac{\max(k^2, \nu^2)}{k}. \quad (40)$$

Thus, for large ν, it is easy to invert the operator on the left-hand side of (39). The only problematic term is the one for which

$k = k_0(\nu)$. We refer to this mode as the critical one, for a fixed value ν.

Our strategy for the rest of the argument is to write, if possible, the resolvent problem (39) in an equivalent form of the type

$$F = \mathcal{R}F + \mathcal{T}G, \tag{41}$$

where the operator $\mathcal{R}$ has small norm in $B(L^2)$ (for large μ) and $\mathcal{T}$ satisfies appropriate bounds in $B(L^2)$. This would allow us to solve (41) via Neumann series and get the bound $\|F\|_{L^2} \le 2\|\mathcal{T}G\|_{L^2}$.

4.2. Equations for the critical and non-critical modes: preliminary steps

We would like to project Eq. (39) on the critical mode P_{k_0}. Recall $P_{k_0}F(x) = \hat{F}(k_0)e^{ik_0x}$, while $F_{\neq k_0} = F - P_{k_0}F$. So, for fixed ν, apply P_{k_0}. Noting that

$$P_{k_0}(VF) = P_{k_0}(VF_{k_0}) + P_{k_0}(VF_{\neq k_0}),$$

we have

$$(-\partial_x^2 - i\mu\partial_x^{-1} - \delta\partial_x^{-1})F_{k_0} - P_{k_0}(VF_{k_0}) = \partial_x^{-1}G_{k_0} + P_{k_0}(VP_{\neq k_0}F). \tag{42}$$

For the non-critical mode, we project with $P_{\neq k_0}$ in (39). We obtain, in a similar way,

$$(-\partial_x^2 - i\mu\partial_x^{-1} - \delta\partial_x^{-1})F_{\neq k_0} - P_{\neq k_0}(VF_{\neq k_0}) = \partial_x^{-1}G_{\neq |k_0|} + P_{\neq k_0}(VP_{k_0}F).$$

We can rewrite this last equation in the following way

$$(-\partial_x^2 - i\mu\partial_x^{-1} - P_{\neq k_0}VP_{\neq k_0})F_{\neq k_0} = \delta\partial_x^{-1}F_{\neq k_0} + \partial_x^{-1}G_{\neq |k_0|} + P_{\neq k_0}(VP_{k_0}F). \tag{43}$$

Consider the self-adjoint operator that arises in the previous calculation

$$\mathcal{M} = \mathcal{M}_{V,\nu} := -\partial_x^2 - i\mu\partial_x^{-1} - P_{\neq k_0}(VP_{\neq k_0}\cdot), D(\mathcal{M}) = H^2,$$

which acts invariantly on the subspace $P_{\neq k_0}[L^2]$. *We claim that for a sufficiently large ν, the operator $\mathcal{M}$ is invertible, in particular $\mathcal{M}^{-1}[P_{\neq k_0}[L^2]] = P_{\neq k_0}[L^2]$.*

In fact, assuming that ν so large that $\nu > C\|V\|_{L^\infty}$, for some absolute constant C, its inverse has favorable $B(L^2)$ bounds, namely

$$\|\mathcal{M}^{-1}\|_{P_{\neq k_0}[L^2]\to P_{\neq k_0}[L^2]} \le C\frac{k}{\max(k^2, \nu^2)} \tag{44}$$

Indeed, this follows by realizing that one can expand $\mathcal{M}^{-1}$ in a Neumann series,

$$\mathcal{M}^{-1} = \sum_{l=0}^{\infty}((-\partial_x^2 - i\mu\partial_x^{-1})^{-1}P_{\neq k_0}(VP_{\neq k_0}\cdot))^l(-\partial_x^2 - i\mu\partial_x^{-1})^{-1}$$

which converges once we take into account the bound (40), which yields

$$\|(-\partial_x^2 - i\mu\partial_x^{-1})^{-1}P_{\neq k_0}(VP_{\neq k_0}\cdot)\|_{B(L^2)} \le C\frac{k}{\max(k^2, \nu^2)}\|V\|_{L^\infty} \le C\nu^{-1}\|V\|_{L^\infty} < \frac{1}{2},$$

under the assumption $\|V\|_{L^\infty} \ll \nu$ made earlier. In addition, estimating again in the Neumann series, we find that $\|\mathcal{M}^{-1}\|_{B(P_{\neq k_0}[L^2])} \le 2\|(-\partial_x^2 - i\mu\partial_x^{-1})^{-1}\|_{B(P_{\neq k_0}[L^2])}$, hence the bound (44).

With (44) in hand, let us proceed with the analysis of the equations of the critical mode. Applying $\mathcal{M}^{-1}$ in (43), we obtain

$$F_{\neq k_0} = \delta\mathcal{M}^{-1}[\partial_x^{-1}F_{\neq k_0}] + \mathcal{M}^{-1}[\partial_x^{-1}G_{\neq k_0}] + \mathcal{M}^{-1}[P_{\neq k_0}[VP_{k_0}F]]. \tag{45}$$

Plugging this back in Eq. (42) and reorganizing terms yields

$$[-\partial_x^2 - i\mu\partial_x^{-1} - P_{k_0}[VP_{k_0}(\cdot)] - P_{k_0}VP_{\neq k_0}\mathcal{M}^{-1}P_{\neq k_0}VP_{k_0}(\cdot)]F_{k_0} = \delta\partial_x^{-1}F_{k_0} + \partial_x^{-1}G_{k_0} + P_{k_0}V(\delta\mathcal{M}^{-1}[\partial_x^{-1}F_{\neq k_0}] + P_{k_0}V\mathcal{M}^{-1}[\partial_x^{-1}G_{\neq k_0}])$$

We now introduce another self-adjoint operator, namely

$$\mathcal{Q} := -\partial_x^2 - i\mu\partial_x^{-1} - P_{k_0}[VP_{k_0}(\cdot)] - P_{k_0}VP_{\neq k_0}\mathcal{M}^{-1}P_{\neq k_0}VP_{k_0}. \tag{46}$$

Note that $\mathcal{Q}$ acts invariantly on the subspace $P_{k_0}[L^2]$. In fact, its action is in the form

$$\mathcal{Q}[\sum_k \hat{f}(k)e^{ikx}] = (k_0^2 - \frac{\mu}{k_0} - c_\nu)\hat{f}(k_0)e^{ik_0x} + \sum_{k\neq k_0}(k^2 - \frac{\mu}{k})\hat{f}(k)e^{ikx}, \tag{47}$$

where

$$c_\nu = \hat{V}(0) + VP_{\neq k_0}\widehat{\mathcal{M}^{-1}}P_{\neq k_0}V(0) = \frac{1}{\sqrt{2\pi}}[\int_{-\pi}^{\pi} V(x)dx + \int_{-\pi}^{\pi} VP_{\neq k_0}\mathcal{M}^{-1}P_{\neq k_0}V(x)dx].$$

Clearly, c_ν is a real constant, satisfying $|c_\nu| \le C\|V\|_{L^\infty}(1+\|V\|_{L^\infty})$.

The operator Q allows us to rewrite the last relation for F_{k_0} in the form,

$$(\mathcal{Q} + i\frac{\delta}{k_0})F_{k_0} = \delta P_{k_0}V\mathcal{M}^{-1}\partial_x^{-1}F_{\neq k_0} + \partial_x^{-1}G_{k_0} + P_{k_0}V\mathcal{M}^{-1}[\partial_x^{-1}G_{\neq k_0}]$$

Recall though that $\mathcal{Q}$ is self-adjoint, so $(\mathcal{Q} + i\frac{\delta}{k_0})$ is invertible, and in fact, from (14), we have the bound

$$\|(\mathcal{Q} + i\frac{\delta}{k_0})^{-1}\|_{B(L^2)} \le \frac{k_0}{\delta} \tag{48}$$

This yields the formula,

$$F_{k_0} = \mathcal{R}_{k_0}F + \mathcal{T}_{k_0}G. \tag{49}$$

where, we have introduced the operators

$$\mathcal{R}_{k_0} := \delta(\mathcal{Q} + i\frac{\delta}{k_0})^{-1}P_{k_0}VP_{\neq k_0}\mathcal{M}^{-1}\partial_x^{-1}P_{\neq k_0}$$

$$\mathcal{T}_{k_0} := (\mathcal{Q} + i\frac{\delta}{k_0})^{-1}\partial_x^{-1}P_{k_0} + (\mathcal{Q} + i\frac{\delta}{k_0})^{-1}P_{k_0}VP_{\neq k_0}\mathcal{M}^{-1}\partial_x^{-1}P_{\neq k_0}.$$

4.3. Construction of the resolvent

We collect our findings so far in the following proposition.

Proposition 3. *The resolvent equation (37) can be equivalently written in the form*

$F_{k_0} = \mathcal{R}_{k_0}F + \mathcal{T}_{k_0}G$, $F_{\neq k_0} = \mathcal{R}_{\neq k_0}F + \mathcal{T}_{\neq k_0}G$, or $F = \mathcal{R}F + \mathcal{T}G$ in short, where

$$\|\mathcal{R}_{k_0}\|_{B(L^2)} \le C\nu^{-1}\|V\|_{L^\infty}, \tag{50}$$

$$\|\mathcal{R}_{\neq k_0}\|_{B(L^2)} \le C\nu^{-1}\|V\|_{L^\infty}, \|\mathcal{T}_{\neq k_0}\|_{B(L^2)} \le C\nu^{-2}. \tag{51}$$

Proof. Most of the statements have been established in the discussion preceding the statement of the Proposition, but we collect all the information herein. For (50), we have by (48) and (44),

$$\|\mathcal{R}_{k_0}\|_{B(L^2)} = \|\delta(\mathcal{Q} + i\frac{\delta}{k_0})^{-1}P_{k_0}VP_{\neq k_0}\mathcal{M}^{-1}\partial_x^{-1}P_{\neq k_0}\|_{B(L^2)} \le C\nu^{-1}\|V\|_{L^\infty}.$$

We can represent $\mathcal{R}_{\neq k_0}$ from (45),

$$\mathcal{R}_{\neq k_0} = (I - \delta\mathcal{M}^{-1}\partial_x^{-1}P_{\neq k_0})^{-1}\mathcal{M}^{-1}P_{\neq k_0}VP_{k_0}, \tag{52}$$

whence the estimate for $\|\mathcal{R}_{\neq k_0}\|_{B(L^2)}$ easily follows. Again from (45),

$$\mathcal{T}_{\neq k_0} = (I - \delta\mathcal{M}^{-1}\partial_x^{-1}P_{\neq k_0})^{-1}\mathcal{M}^{-1}\partial_x^{-1}P_{\neq k_0}, \tag{53}$$

whence the estimate $\|\mathcal{T}_{\neq k_0}\|_{B(L^2)} \le C\nu^{-2}$ follows from (44) as well. □

4.4. Proof of Theorem 3

We present the proof of Theorem 3 as a direct corollary of Proposition 3. First, for the zero mode, we use the relation (38), which yields

$|\hat{F}(0)| \leq C\mu^{-1}|\hat{G}(0)|$. Next, we assume, without loss of generality that $\hat{F}(0) = \hat{G}(0) = 0$.

Assume that $\mu \gg \max(\|V\|_{L^\infty}^3, 1)$. Then, the estimates for $\mathcal{R}$, namely

$$\|\mathcal{R}\|_{B(L^2)} \leq C\nu^{-1}\|V\|_{L^\infty} \leq C\mu^{-1/3}\|V\|_{L^\infty} \ll 1,$$

guarantee that $Id - \mathcal{R}$ is invertible on L^2, and in fact $\|(I - \mathcal{R})^{-1}\|_{B(L^2)} < \frac{1}{2}$. Thus, we can resolve the resolvent equation $F = \mathcal{R}F + \mathcal{T}G$ as follows $F = (I - \mathcal{R})^{-1}\mathcal{T}G$. Then, we apply the estimates for $\mathcal{T}$ found in Proposition 3, to obtain

$$\|F\|_{L^2} \leq \|\mathcal{T}_{\neq k_0(\nu)}G\|_{L^2} + \|\mathcal{T}_{k_0(\nu)}G\|_{L^2}.$$

The estimate $\|\mathcal{T}_{\neq k_0(\nu)}G\|_{L^2} \leq C\|G\|_{L^2}$ is in (51), whereas by a direct estimation (using (44))

$$\|\mathcal{T}_{k_0(\nu)}G\|_{L^2} \leq C\frac{k_0}{\delta}(\frac{1}{k_0} + \frac{\|V\|_{L^\infty}}{k_0})\|G\|_{L^2} \leq C\delta^{-1}\|G\|_{L^2}.$$

All in all, this implies that $\|F\|_{L^2} \leq C\delta^{-1}\|G\|_{L^2}$. This shows that the resolvent at $\delta + i\mu$ indeed exists, i.e. it is bounded operator on L^2, whenever $\mu \gg \max(\|V\|_{L^\infty}^3, 1)$. In fact, we have the resolvent estimate

$$\sup_{\mu \in \mathbf{R}} \|\partial_x(-\partial_x^2 - V) - (\delta + i\mu)\|_{B(L^2)} \leq C\delta^{-1}. \tag{54}$$

Note that this estimate blows up as $\delta \to 0$, so establishing Gomilko type bounds (i./e. in the form (24)) for the operator $\partial_x(-\partial_x^2 - V)$ is more subtle than (54).

5. Uniform L^2 bounds for the NLS semigroup

In this section, after appropriate reductions, we eventually reduce matters to the verification of the Gomilko's criteria. Let us work on the reductions first.

5.1. The semigroup $e^{t\mathcal{J}\mathcal{L}}$ grows at most sub-exponentially

The first result is preliminary.

Proposition 4. *Assume that the generator $\mathcal{J}\mathcal{L}$ is spectrally stable, that is $\sigma(\mathcal{J}\mathcal{L}) \subset i\mathbf{R}$. Then, for every $\delta > 0$, there is C_δ, so that for every $t > 0$,*

$$\|e^{t\mathcal{J}\mathcal{L}}f\|_{L^2} \leq C_\delta e^{\delta t}\|f\|_{L^2}. \tag{55}$$

Remark. It suffices to assume that there is no spectrum in the set $\{z : \Re z \neq 0, |\Im z| \leq 2\|V\|_{L^\infty}\}$, since the remaining spectrum is guaranteed to be on $i\mathbf{R}$ by Proposition 2.

Proof. Let $\delta_0 > 0$. We apply the Gearhart–Prüss theorem to the semi-group generated by $\mathcal{J}\mathcal{L} - \delta_0 I$. Indeed, if we show that $e^{t(\mathcal{J}\mathcal{L} - \delta_0)}$ has negative growth bound, then in particular,

$$\|e^{t\mathcal{J}\mathcal{L}}f\|_{L^2} \leq C_{\delta_0}e^{\delta_0 t}\|f\|_{L^2}.$$

As such an inequality holds true for all $\delta_0 > 0$, the Proposition follows. According to Gearhart–Prüss, and the assumptions, it thus remains to show

$$\sup_{\mu \in \mathbf{R}} \|(\mathcal{J}\mathcal{L} - \delta_0 + i\mu)^{-1}\|_{L^2 \to L^2} \leq C_{\delta_0}. \tag{56}$$

First, observe that we are assuming that $\mathbf{C}_+ \subset \rho(\mathcal{J}\mathcal{L})$. So, the $B(L^2)$ valued function $z \to (\mathcal{J}\mathcal{L} - z)^{-1}$ is holomorphic on $\mathbf{C}_+$. In particular, it is continuous, and hence bounded on the compact subsets, say on $K = \{z : z = \delta_0 - i\mu : |\mu| \leq 4\|V\|_{L^\infty}\}$. It follows that

$$\sup_{|\mu| \leq 4\|V\|_{L^\infty}} \|(\mathcal{J}\mathcal{L} - \delta_0 + i\mu)^{-1}\|_{L^2 \to L^2} \leq C_{\delta_0, V}.$$

For $\mu : |\mu| \geq 4\|V\|_{L^\infty}$, we use the bounds from Proposition 2, with $\delta = \delta_0$. Namely, (35) and the subsequent bound for z_2 are in the form

$$\|z_1\|_{L^2} + \|z_2\|_{L^2} \leq C_{\delta_0}(1 + \mu^{-1})(\|f_1\|_{L^2} + \|f_2\|_{L^2}),$$

when $|\mu| \geq 4\|V\|_{L^\infty}$. Thus,

$$\sup_{|\mu| \geq 4\|V\|_{L^\infty}} \|(\mathcal{J}\mathcal{L} - \delta_0 + i\mu)^{-1}\|_{L^2 \to L^2} \leq C_{\delta_0}.$$

This verifies the Gearhart–Prüss criteria, and hence the sub-exponential bounds are established. □

Note: Under the spectral stability assumption $\sigma(\mathcal{J}\mathcal{L}) \subset i\mathbf{R}$, as a simple consequence of Proposition 4, we can in fact bound the Gomilko's quantities, if $\delta \geq \delta_0 > 0$. More precisely, we claim that (55) implies

$$\sup_{\delta \geq \delta_0} \delta \int_{-\infty}^{\infty} [\|(\mathcal{J}\mathcal{L} - (\delta + i\mu))^{-1}f\|_{L^2}^2 + \|(\mathcal{L}\mathcal{J} + (\delta + i\mu))^{-1}f\|_{L^2}^2] d\mu \leq C_{\delta_0}\|f\|_{L^2}^2. \tag{57}$$

Indeed, (55) establishes negative growth rate for the semigroup generated by $\mathcal{A} = \mathcal{J}\mathcal{L} - \delta$ for any δ. In particular, the semigroup is uniform in time for $\delta = \frac{\delta_0}{2}$ and hence, the necessity statement in Theorem 7 applies. Thus,

$$\begin{aligned}
&\sup_{\sigma > 0} \sigma \int_{-\infty}^{\infty} [\|(\mathcal{J}\mathcal{L} - \frac{\delta_0}{2} - (\sigma + i\mu))^{-1}f\|_{L^2}^2 \\
&\quad + \|(\mathcal{L}\mathcal{J} + \frac{\delta_0}{2} + (\sigma + i\mu))^{-1}f\|_{L^2}^2] d\mu \\
&\leq C_{\delta_0}\|f\|_{L^2}^2.
\end{aligned}$$

In particular, for $\sigma > \frac{\delta_0}{2}$, the last estimate implies that we have control in the form

$$\sup_{\delta \geq \delta_0} \delta \int_{-\infty}^{\infty} [\|(\mathcal{J}\mathcal{L} - (\delta + i\mu))^{-1}f\|_{L^2}^2 + \|(\mathcal{L}\mathcal{J} + (\delta + i\mu))^{-1}f\|_{L^2}^2] d\mu \leq 2C_{\delta_0}\|f\|_{L^2}^2.$$

This last inequality does not of course imply the full Gomilko sufficient condition (24), but it shows that it remains to control

$$\limsup_{\delta \to 0+} \delta \int_{-\infty}^{\infty} [\|(\mathcal{J}\mathcal{L} - (\delta + i\mu))^{-1}f\|_{L^2}^2 + \|(\mathcal{L}\mathcal{J} + (\delta + i\mu))^{-1}f\|_{L^2}^2] d\mu \leq C\|f\|_{L^2}^2.$$

This is what we do next.

5.2. Proof of Theorem 2

We need to show the following estimate for the solutions z_1, z_2 of (32), (33),

$$\int_{-\infty}^{\infty} [\|z_1(\mu)\|^2 + \|z_2(\mu)\|_{L^2}^2] d\mu \leq C\delta^{-1}[\|f_1\|^2 + \|f_2\|^2]. \tag{58}$$

for all sufficiently small $\delta > 0$, say $0 < \delta < 1$, and an absolute constant, independent of δ. As we have mentioned before, the cases $\mu > 0$ and $\mu < 0$ are symmetric, so it suffices to consider just $\mu > 0$. In accordance with the results of Proposition 2, we split the integral: $\mu > \max(2\|V\|_{L^\infty}, 2)$ and $0 < \mu \leq \max(2\|V\|_{L^\infty}, 2)$. Historically, at least in quantum mechanical contexts, spectral parameters have played a role of an appropriate energy levels of the corresponding atoms, so we adopt these notations and refer to these two cases as high energies and low energies.

5.2.1. The case of high energies

Let us show that the estimate for z_2 reduces to the estimate for z_1. We have

$$\int_{\max(2\|V\|_{L^\infty},2)}^{\infty} \|z_2(\mu)\|_{L^2}^2 d\mu \le 2\int_{\max(2\|V\|_{L^\infty},2)}^{\infty} \|(\mathcal{H}_0+\mu-i\delta)^{-1} f_2\|_{L^2}^2 d\mu +$$

$$f\ +2\int_{\max(2\|V\|_{L^\infty},2)}^{\infty} \|(\mathcal{H}_0+\mu-i\delta)^{-1} V z_1(\mu)\|_{L^2}^2 d\mu$$

The first integral above is controlled by $C\delta^{-1}\|f_2\|_{L^2}^2$, by Lemma 1, whereas the standard estimate $\|(\mathcal{H}_0+\mu-i\delta)^{-1}\|_{B(L^2)} \le C\mu^{-1}$ (when $\mu > 2\|V\|_{L^\infty}$), yields an estimate for the second term in the form

$$C\int_{|\mu|>\max(2\|V\|_{L^\infty},2)} \mu^{-2}\|V\|_{L^\infty}^2 \|z_1(\mu)\|_{L^2}^2 d\mu.$$

Thus, an estimate in the form

$$\int_{\max(2\|V\|_{L^\infty},2)} \|z_1(\mu)\|^2 d\mu \le C\delta^{-1}[\|f_1\|^2+\|f_2\|^2]. \tag{59}$$

would imply the estimate for z_2 as well as the required estimate for z_1. Thus, we have reduced matters in to the verification of estimate (59). We henceforth concentrate on showing (59). Recall that

$$\|(I-i\delta(\mathcal{H}-\mu+i\delta)^{-1}V(\mathcal{H}_0+\mu-i\delta)^{-1}(\mathcal{H}_0+\mu)^{-1}V)^{-1}\|_{B(L^2)} \le 2,$$

whenever $\mu > 2\|V\|_{L^\infty}$. So, (59) reduces to

$$\int_{\max(2\|V\|_{L^\infty},2)} \|(\mathcal{H}-\mu+i\delta)^{-1}V(\mathcal{H}_0+\mu-i\delta)^{-1}f\|^2 d\mu \le C\delta^{-1}\|f\|_{L^2}^2 \tag{60}$$

since the required estimate for the f_1 term is

$$\int_{\max(2\|V\|_{L^\infty},2)} \|(\mathcal{H}-\mu+i\delta)^{-1}f\|^2 d\mu \le C\delta^{-1}\|f\|_{L^2}^2$$

follows from Lemma 1.

Before we present the further details, let us point out that $\mathcal{H}$ has a finite number of eigenvalues (say μ_j, with eigenvectors e_j) in each compact interval. So,

$$\int_{\max(2\|V\|_{L^\infty},2)} \|(\mathcal{H}-\mu+i\delta)^{-1}V(\mathcal{H}_0+\mu-i\delta)^{-1}f\|^2 \le$$

$$\le \sum_{n=[\max(2\|V\|_{L^\infty},2)]}^{\infty} \int_n^{n+1} \|(\mathcal{H}-\mu+i\delta)^{-1}V(\mathcal{H}_0+\mu-i\delta)^{-1}f\|^2 \le$$

$$\le \sum_{n=[\max(2\|V\|_{L^\infty},2)]}^{\infty} \sum_{j:\mu_j\in[n-1,n+2]} \int_n^{n+1} |\mu_j-\mu+i\delta|^{-2}$$

$$\times\ |\langle V(\mathcal{H}_0+\mu-i\delta)^{-1}f, e_j\rangle|^2 d\mu +$$

$$+\sum_{n=[\max(2\|V\|_{L^\infty},2)]}^{\infty} \int_n^{n+1} \|(P_{\le n-1}+P_{\ge n+2})[V(\mathcal{H}_0+\mu-i\delta)^{-1}f]\|_{L^2}^2 d\mu.$$

The second term is easier to control. We have

$$\sum_{n=[\max(2\|V\|_{L^\infty},2)]}^{\infty} \int_n^{n+1} \|(P_{\le n-1}+P_{\ge n+2})[V(\mathcal{H}_0+\mu-i\delta)^{-1}f]\|_{L^2}^2 d\mu$$

$$\le C\|V\|_{L^\infty}^2 \sum_{n=[\max(2\|V\|_{L^\infty},2)]}^{\infty} \int_n^{n+1} \|(\mathcal{H}_0+\mu-i\delta)^{-1}f\|_{L^2}^2 d\mu$$

$$\le C\|V\|_{L^\infty}^2 \int_{-\infty}^{\infty} \|(\mathcal{H}_0+\mu-i\delta)^{-1}f\|_{L^2}^2 \le C\delta^{-1}\|V\|_{L^\infty}^2\|f\|_{L^2}^2,$$

where in the last step, we have used Lemma 1.

In order to control

$$\sum_{n=[\max(2\|V\|_{L^\infty},2)]}^{\infty} \sum_{j:\mu_j\in[n-1,n+2]} \int_n^{n+1} |\mu_j-\mu+i\delta|^{-2}$$

$$\times\ |\langle V(\mathcal{H}_0+\mu-i\delta)^{-1}f, e_j\rangle|^2 d\mu,$$

we need to expand, via the resolvent identity and for $n \le \mu \le n+1$,

$$(\mathcal{H}_0+\mu-i\delta)^{-1} = \sum_{l=0}^{\infty}(n-\mu)^l[(\mathcal{H}_0+n-i\delta)^{-1}]^{l+1}.$$

Note that this is convergent, due to the fact that $|n-\mu| \le 1$, while $\|(\mathcal{H}_0+n-i\delta)^{-1}\|_{B(L^2)} \le Cn^{-1}$. It follows by Cauchy–Schwartz's inequality that

$$|\langle V(\mathcal{H}_0+\mu-i\delta)^{-1}f, e_j\rangle|^2 = |\langle V\sum_{l=0}^{\infty}(n-\mu)^l[(\mathcal{H}_0+n-i\delta)^{-1}]^{l+1}f, e_j\rangle|^2$$

$$\le \left(\sum_{l=0}^{\infty}|\langle V[(\mathcal{H}_0+n-i\delta)^{-1}]^{l+1}f, e_j\rangle|\right)^2$$

$$\le \sum_{l=0}^{\infty}(1+l^2)|\langle V[(\mathcal{H}_0+n-i\delta)^{-1}]^{l+1}f, e_j\rangle|^2 \sum_{l=0}^{\infty}(1+l^2)^{-1}.$$

Denote $F_{n,l} := V[(\mathcal{H}_0+n-i\delta)^{-1}]^{l+1}f$, an L^2 function, which is independent on μ, j. Note however,

$$\|F_{n,l}\|_{L^2} \le \frac{\|V\|_{L^\infty}}{n^{l+1}}\|f\|_{L^2}.$$

We now need to estimate

$$\sum_{l\ge 0}(1+l^2) \sum_{n=[\max(2\|V\|_{L^\infty},2)]}^{\infty} \sum_{\mu_j\in[n-1,n+2]} |\langle F_{n,l}, e_j\rangle|^2 \int_n^{n+1} |\mu_j-\mu+i\delta|^{-2} d\mu. \tag{61}$$

For each particular $j : \mu_j \in (n-1, n+2)$,

$$\int_n^{n+1} |\mu_j-\mu+i\delta|^{-2} d\mu \le \int_0^{\infty} \frac{1}{(\mu_j-\mu)^2+\delta^2} d\mu = \frac{\pi}{2}\delta^{-1}$$

Going back to (61), we can estimate it by

$$C\delta^{-1}\sum_{l\ge 0}(1+l^2) \sum_{n=[\max(2\|V\|_{L^\infty},2)]}^{\infty} \sum_{\mu_j\in[n-1,n+2]} |\langle F_{n,l}, e_j\rangle|^2 \le$$

$$\le C\delta^{-1}\sum_{l\ge 0}(1+l^2) \sum_{n=[\max(2\|V\|_{L^\infty},2)]}^{\infty} \|F_{n,l}\|_{L^2}^2 \le$$

$$\le C\delta^{-1}\|V\|_{L^\infty}^2\|f\|_{L^2}^2 \sum_{l\ge 0}(1+l^2) \sum_{n=[\max(2\|V\|_{L^\infty},2)]}^{\infty} \frac{1}{n^{2l+2}} \le$$

$$\le C\delta^{-1}\|V\|_{L^\infty}^2\|f\|_{L^2}^2 \sum_{l\ge 0}\frac{(1+l^2)}{2^{2l+1}} \le C\delta^{-1}\|f\|_{L^2}^2.$$

This establishes (60) and hence the case $\mu > \max(2\|V\|_{L^\infty}, 2)$ is analyzed in full.

5.2.2. Low energies estimate

For this step, we proceed as follows. Fix a large real number, $N : N > \max(2\|V\|_{L^\infty}, 2)$. Note that by assumption $\sigma(\mathcal{JL}) \subset i\mathbf{R}$. Moreover, since $\sigma(\mathcal{JL})$ consists of isolated eigenvalues, we can further select N, so that $\pm iN$ are not eigenvalues. Finally, by Hamiltonian symmetry, all the eigenvalues of $\mathcal{JL}$ inside the interval $[-iN, iN]$ are in the form $\pm i\mu_j, j = 1, \dots, J_N$, where $0 \le \mu_j < N$. For each $\pm i\mu_j$, consider the Riesz projection P_j that is

$$P_j := \frac{1}{2\pi i}\int_{\gamma_j} (\mathcal{J}\mathcal{L}-z)^{-1}dz,$$

where γ_j is a positively oriented closed curve of index one that encloses both $\pm i\mu_j$, but no other spectrum of $\mathcal{J}\mathcal{L}$. The operators $P_j\mathcal{J}\mathcal{L} = P_j\mathcal{J}\mathcal{L}P_j, j = 1,\dots,J_N$ can be represented as finite dimensional matrices of dimension n_j, which consists of l_j separate Jordan blocks, each of dimension $n_j^l, l = 1,\dots,l_j$. For these types of matrices, it is well-known that

$$\|P_j e^{t\mathcal{J}\mathcal{L}}\|_{B(L^2)} \leq Ct^{n_j^{l_j}-1}. \tag{62}$$

Introduce then the Riesz projections $P_N := \sum_{j=1}^{J_N} P_j$ and $Q_N = I - P_N$. Clearly,

$$\|e^{tP_N\mathcal{J}\mathcal{L}}\|_{B(L^2)} = \|P_N e^{t\mathcal{J}\mathcal{L}}\|_{B(L^2)} \leq Ct^{\max_{j\in(1,J_N)} n_j^{l_j}-1}. \tag{63}$$

On the other hand, the operator $Q_N\mathcal{J}\mathcal{L} : Q_N(L^2) \to Q_N(L^2)$ has no spectrum in a neighborhood of $[-iN, iN]$. Thus, its resolvent $z \to (Q_N\mathcal{J}\mathcal{L}-z)^{-1}$ is analytic, $B(Q_N(L^2))$ valued function in such a neighborhood, so by its continuity

$$\lim_{\delta\to 0+}\|(\mathcal{J}\mathcal{L}\pm(\delta+i\mu))^{-1}-(\mathcal{J}\mathcal{L}\pm i\mu)^{-1}\|_{B(Q_N(L^2))} = 0.$$

This implies that for $f = Q_N f$,

$$\limsup_{\delta\to 0+}\int_{-N}^{N}\|(\mathcal{J}\mathcal{L}\pm(\delta+i\mu))^{-1}f\|_{L^2}^2 d\mu = \int_{-N}^{N}\|(\mathcal{J}\mathcal{L}\pm i\mu)^{-1}f\|_{L^2}^2 d\mu \leq C_N\|f\|_{L^2}^2. \tag{64}$$

Thus, for $f = Q_N f$,

$$\limsup_{\delta\to 0+}\delta\int_{-N}^{N}\|(\mathcal{J}\mathcal{L}\pm(\delta+i\mu))^{-1}f\|_{L^2}^2 d\mu = 0.$$

Since we have dealt with the high energies case before, we have that for all $f \in L^2$, so in particular for $f \in Q_N(L^2)$

$$\limsup_{\delta\to 0+}\delta\int_{|\mu|>\max(4\|V\|_{L^\infty},2)}\|(\mathcal{J}\mathcal{L}\pm(\delta+i\mu))^{-1}f\|_{L^2}^2 d\mu \leq C\|f\|_{L^2}$$

The selection of N ensures that we have covered the whole real axes, once we combine the last two estimates. Thus, for $f \in Q_N(L^2)$

$$\limsup_{\delta\to 0+}\delta\int_{-\infty}^{\infty}\|(\mathcal{J}\mathcal{L}\pm(\delta+i\mu))^{-1}f\|_{L^2}^2 d\mu \leq C\|f\|_{L^2}.$$

Applying the Gomilko's sufficient condition to the Hilbert subspace $Q_N(L^2)$, on which the semigroup $e^{t\mathcal{J}\mathcal{L}}$ acts invariantly, we conclude

$$\sup_{0<t<\infty}\|e^{t\mathcal{J}\mathcal{L}}Q_N f\|_{L^2} \leq C\|f\|_{L^2}.$$

The complementary estimate is provided in (63). All in all,

$$\|e^{t\mathcal{J}\mathcal{L}}f\|_{L^2} \leq Ct^{\max_{j\in(1,J_N)} n_j^{l_j}-1}\|f\|_{L^2}. \tag{65}$$

6. Uniform L^2 bounds for the KdV semigroup: Proof of Theorem 4

We are approaching the problem in the same way as its NLS counterpart. The first step is to realize that the assumption $\sigma(\partial_x(-\partial_x^2 - V)) \subset i\mathbf{R}$, together with the estimate (54), which shows the uniform bound $\|(\partial_x(-\partial_x^2-V)-(\delta+i\mu))^{-1}\|_{B(L^2)} \leq C\delta^{-1}$ for all $\mu \gg \max(\|V\|_{L^\infty},1)^3$, implies that for all $\delta > 0$, there exists a constant C_δ, so that

$$\sup_{\mu\in\mathbf{R}}\|(\partial_x(-\partial_x^2 - V)-(\delta+i\mu))^{-1}\|_{B(L^2)} \leq C_\delta.$$

Indeed, for large energies, this is just (54), while for low energies, we just exploit the analyticity and hence the continuity of the map $\mu \to (\partial_x(-\partial_x^2 - V)-(\delta+i\mu))^{-1}$ on compact intervals $\mu \in (-N, N), N \sim \max(\|V\|_{L^\infty},1)^3$. Based on this uniform bound for the resolvent on each vertical line $\{z : \Re z = \delta\}$ one can infer sub-exponential growth of the semi-group $e^{t\partial_x(-\partial_x^2-V)}$. That is $\|e^{t\partial_x(-\partial_x^2-V)}\|_{B(L^2)} \leq C_\kappa e^{\kappa t}$ for each $\kappa > 0$.

As in the NLS case, we will show uniform L^2 bounds on the high energies, the low energies are treated in an identical way. By Gomilko's theorem, this reduces to the estimates

$$\delta\int_{|\mu|>N^3}\|(\partial_x(-\partial_x^2 - V)-(\delta+i\mu))^{-1}G\|_{L^2}^2 d\mu \leq C\|G\|_{L^2} \tag{66}$$

$$\delta\int_{|\mu|>N^3}^{\infty}\|((-\partial_x^2 - V)\partial_x-(\delta+i\mu))^{-1}G\|_{L^2}^2 d\mu \leq C\|G\|_{L^2} \tag{67}$$

for $N \gg \max(\|V\|_{L^\infty},1)$. As in the NLS case, the sub-exponential bound implies that (see the proof of (57), based on the estimate (55)) for each $\delta_0 > 0$, there is C_{δ_0}, so that

$$\sup_{\delta\geq\delta_0}\delta\int_{|\mu|>N^3}\|(\partial_x(-\partial_x^2 - V)-(\delta+i\mu))^{-1}G\|_{L^2}^2 d\mu \leq C_{\delta_0}\|G\|_{L^2},$$

similar for (67). Thus, matters are reduced to the case $\delta : 0 < \delta < 1$, which we assume henceforth.

We start with the proof of (66), the proof of (67) goes through much of the same estimates, with some extra complications, which will be addressed later on.

6.1. Proof of (66)

The cases $\mu > 0$ and $\mu < 0$ are symmetric, so we take $\mu > 0$. The first thing to observe is that by (38), we clearly have

$$\int_1^\infty|\hat{F}(0)|^2 d\mu \leq C\int_1^\infty|\hat{G}(0)|^2\mu^{-2}d\mu \leq C\|G\|_{L^2}^2 \leq C\delta^{-1}\|G\|_{L^2}^2,$$

since $\delta < 1$. As we have observed earlier, we may henceforth assume $\hat{F}(0) = \hat{G}(0) = 0$. Next, we perform change of variables $\mu = \nu^3$ in (66). We need to control

$$\delta\int_N^\infty\|(\partial_x(-\partial_x^2 - V)-(\delta+i\nu^3))^{-1}G\|_{L^2}^2\nu^2 d\nu.$$

Based on the representation in Proposition 3, we can control the contributions of the terms $(I-\mathcal{R})^{-1}\mathcal{T}_{\neq k_0(\nu)}, (I-\mathcal{R})^{-1}\tilde{\mathcal{T}}_{k_0}$ as follows – since $\|(I-\mathcal{R})^{-1}\|_{L^2\to L^2} \leq \frac{1}{2}$, it suffices to control

$$\int_N^\infty\|\mathcal{T}_{\neq k_0(\nu)}G\|_{L^2}^2\nu^2 d\nu \leq C\int_N^\infty\nu^{-2}\|G\|_{L^2}^2 d\nu \leq C\|G\|_{L^2}^2 \leq C\delta^{-1}\|G\|_{L^2}^2,$$

due to the bound (50) and $\delta < 1$. For the remaining term,

$$\int_N^\infty\|\mathcal{T}_{k_0(\nu)}G\|_{L^2}^2\nu^2 d\nu, \tag{68}$$

observe that the operator involves $(\mathcal{Q}+i\frac{\delta}{k_0})^{-1}P_{k_0}$, has a very simple form according to (47), namely

$$(\mathcal{Q}+i\frac{\delta}{k_0})^{-1}P_{k_0}f(x) = \frac{1}{k_0^2-\frac{\nu^3}{k_0}-c_\nu+i\frac{\delta}{k_0}}\hat{f}(k_0)e^{ik_0x}.$$

In addition, let us analyze the real part of the modified dispersion term. More precisely, introducing $\tilde{k} : \tilde{k}^3 = k_0^3 - c_\nu k_0$, so that $\tilde{k} = k_0(\nu) + O(\nu^{-1})$ (recall $c_\nu = O(1)$), we write

$$k_0^2 - \frac{\nu^3}{k_0} - c_\nu = \frac{\tilde{k}^3-\nu^3}{k_0} = (\tilde{k}-\nu)\frac{\tilde{k}^2+\tilde{k}\nu+\nu^2}{k_0} \sim (\tilde{k}-\nu)\nu.$$

We now estimate the two terms arising in $\mathcal{T}_{k_0(\nu)}G$, namely

$$\int_N^\infty\|(\mathcal{Q}+i\frac{\delta}{k_0})^{-1}\partial_x^{-1}G_{k_0}\|_{L^2}^2\nu^2 d\nu \leq C\sum_{l=N}^\infty\int_{l-1/2}^{l+1/2}\|(\mathcal{Q}+i\frac{\delta}{l})^{-1}G_l\|_{L^2}^2 d\nu$$

where we have observed that for $\nu \in (l-1/2, l+1/2)$, $k_0(\nu) = l$. Next, we partition each of the intervals $(l-1/2, l+1/2)$ as follows

$$(l-1/2, l+1/2) \subset \{\nu : |\nu - \tilde{k}| < \delta l^{-2}\} \cup \cup_{m=1}^{\infty} \times \{\nu : |\nu - \tilde{k}| \sim 2^m \delta l^{-2}\} =: \mathcal{A}_0 \cup_{m=1}^{\infty} \mathcal{A}_m.$$

Note that $\nu \in \mathcal{A}_0$ implies (by estimating with the imaginary part), $\frac{1}{|k_0^2 - \frac{\nu^3}{k_0} + c_\nu + i\frac{\delta}{k_0}|} \sim l\delta^{-1}$, while $\nu \in \mathcal{A}_m$ gives $\frac{1}{|k_0^2 - \frac{\nu^3}{k_0} + c_\nu + i\frac{\delta}{k_0}|} \sim 2^{-m} l\delta^{-1}$. Taking into account that $|\mathcal{A}_m| \sim 2^m \delta l^{-2}$,

$$\int_{l-1/2}^{l+1/2} \|(\mathcal{Q} + i\frac{\delta}{l})^{-1} G_l\|_{L^2}^2 d\nu \le C \sum_{m=0}^{\infty} 2^{-2m} l^2 \delta^{-2} \|G_l\|_{L^2}^2 \int_{\mathcal{A}_m} d\nu$$
$$\le C\delta^{-1} \|G_l\|_{L^2}^2 \sum_{m=0}^{\infty} 2^{-m},$$

which implies the desired bound, since $\sum_l \|G_l\|_{L^2}^2 \le \|G\|_{L^2}^2$.

For the other term, it is actually reducible to the one that we just handled. Indeed, we need to control

$$\int_N^\infty \|(\mathcal{Q} + i\frac{\delta}{k_0})^{-1} P_{k_0} V P_{\neq k_0} \mathcal{M}^{-1} \partial_x^{-1} G_{\neq k_0}\|_{L^2}^2 \nu^2 d\nu$$

Denoting $\tilde{G} := V P_{\neq k_0} \mathcal{M}^{-1} \partial_x^{-1} G_{\neq k_0}$ (note that $\tilde{G}$ depends on ν and ∂_x^{-1} in front of it is missing), we have, by performing the same steps as above

$$\int_N^\infty \|(\mathcal{Q} + i\frac{\delta}{k_0})^{-1} \tilde{G}_{k_0}\|_{L^2}^2 \nu^2 d\nu \le \sum_{l=N}^{\infty} \sum_{m=0}^{\infty} 2^{-2m} l^4 \delta^{-2} \int_{\mathcal{A}_m} \|\tilde{G}_l(\nu)\|_{L^2}^2 d\nu.$$

According to (44) however, $\|\tilde{G}_l\|_{L^2} \le C\|V\|_{L^\infty} \nu^{-2}$. Plugging this in the previous estimate, together with $|\mathcal{A}_m| \sim 2^m \delta l^{-2}$ yields again the bound $C\delta^{-1}\|G\|_{L^2}^2$.

6.2. Proof of (67)

Write the resolvent equation corresponding to (67) in the form

$$(-\partial_x^2 - V)\partial_x F - (\delta + i\mu)F = G. \tag{69}$$

As in the proof of (66), we start with the contribution of $\hat{F}(0)$. Taking an integral $x \in [-\pi, \pi]$ in (69), we obtain the relation

$$(\delta + i\mu)\hat{F}(0) = -\hat{G}(0) - \widehat{VF}'(0).$$

Clearly, integration by parts implies $|\widehat{VF}'(0)| \le \|V'\|_{L^2} \|F - (2\pi)^{-1} \int_{-\pi}^{\pi} F(y) dy\|_{L^2}$. Hence

$$\int_1^\infty |\hat{F}(0)|^2 d\mu \le C\|G\|_{L^2}^2 + C\|V'\|_{L^2}^2 \int_1^\infty \|F_{\neq 0}\|_{L^2}^2 d\mu,$$

whence the estimate (67) reduces to the control of $\int_1^\infty \|F_{\neq 0}\|_{L^2}^2 d\mu$ in terms of $C\|G\|_{L^2}^2$. That is, without loss of generality, we may and do assume $\hat{F}(0) = \hat{G}(0) = 0$. Applying the change of variables $f = \partial_x F, g = \partial_x G$ in (69), we obtain

$$(-\partial_x^2 - V)f - (\delta + i\mu)\partial_x^{-1} f = \partial_x^{-1} g$$

This last relation is nothing but (39), whence we can resolve it according to Proposition 3 in the following way $f = \mathcal{R}f + \mathcal{T}g$. Taking into account $f = \partial_x F, g = \partial_x G$ and applying ∂_x^{-1} judiciously, we arrive at

$$(I - \partial_x^{-1}\mathcal{R}\partial_x)F = \partial_x^{-1}\mathcal{T}\partial_x G. \tag{70}$$

In order to analyze (70), we need the following lemma.

Lemma 2. *Let $\delta \in (0, 1)$. Then,*

$$\|\partial_x^{-1}\mathcal{R}\partial_x\|_{B(L^2)} \le C\nu^{-1}\|V\|_{L^\infty}. \tag{71}$$

In particular, if $\nu \gg \|V\|_{L^\infty}$, $(I - \partial_x^{-1}\mathcal{R}\partial_x)$ is invertible and $\|(I - \partial_x^{-1}\mathcal{R}\partial_x)^{-1}\|_{B(L^2)} < \frac{1}{2}$. In addition,

$$\|\partial_x^{-1}\mathcal{T}_{\neq k_0}\partial_x\|_{B(L^2)} \le C\nu^{-2}. \tag{72}$$

Proof. We start with $\partial_x^{-1}\mathcal{R}_{\neq k_0}\partial_x$. This is represented in (52). According to it, by using the von Neumann series,

$$\begin{aligned}\partial_x^{-1}\mathcal{R}_{\neq k_0}\partial_x &= \partial_x^{-1}(I - \delta\mathcal{M}^{-1}\partial_x^{-1}P_{\neq k_0})^{-1}\mathcal{M}^{-1}P_{\neq k_0}VP_{k_0}\partial_x \\ &= \partial_x^{-1}\mathcal{M}^{-1}P_{\neq k_0}VP_{k_0}\partial_x + \\ &+ \sum_{l=1}^{\infty} \partial_x^{-1}(\delta\mathcal{M}^{-1}\partial_x^{-1}P_{\neq k_0})^l \mathcal{M}^{-1}P_{\neq k_0}VP_{k_0}\partial_x.\end{aligned}$$

Using the estimate (44) and $0 < \delta < 1$, we can bound favorably the terms in the sum as follows

$$\|\partial_x^{-1}(\delta\mathcal{M}^{-1}\partial_x^{-1}P_{\neq k_0})^l\mathcal{M}^{-1}P_{\neq k_0}VP_{k_0}\partial_x\|_{B(L^2)} \le C\nu^{-2l}\|V\|_{L^\infty},$$

while

$$\|\partial_x^{-1}\mathcal{M}^{-1}P_{\neq k_0}VP_{k_0}\partial_x\|_{B(L^2)} \le C\nu^{-1}\|V\|_{L^\infty}.$$

As before, this guarantees that if $\nu \gg \|V\|_{L^\infty}$, we have the bound (71). Similarly, using the bound (48), we can estimate $\|\partial_x^{-1}\mathcal{R}_{k_0}\partial_x\|_{B(L^2)} \le C\nu^{-1}\|V\|_{L^\infty}$. Thus, (70) follows by adding the two terms in $\partial_x^{-1}\mathcal{R}\partial_x$.

Regarding $\partial_x^{-1}\mathcal{T}_{\neq k_0}\partial_x$, we have the bound (expanding as in the analysis for $\partial_x^{-1}\mathcal{R}\partial_x$),

$$\|\partial_x^{-1}\mathcal{T}_{\neq k_0}\partial_x\|_{B(L^2)} \le C\nu^{-2}. \quad \square$$

Going back to the required estimate (67), we have that according to (70) and for $\nu : \nu \gg \|V\|_{L^\infty}$, $F = (I - \partial_x^{-1}\mathcal{R}\partial_x)^{-1}\partial_x^{-1}\mathcal{T}\partial_x G$ and $\|F\|_{L^2} \le C\|\partial_x^{-1}\mathcal{T}\partial_x G\|_{L^2}$. Thus, we need to control $\int_N^\infty \|\partial_x^{-1}\mathcal{T}\partial_x G\|^2 \nu^2 d\nu$. Due to the bound (72), we have

$$\int_N^\infty \|\partial_x^{-1}\mathcal{T}_{\neq k_0(\nu)}\partial_x G\|_{L^2}^2 \nu^2 d\nu \le C\int_N^\infty \nu^{-2}\|G\|_{L^2}^2 d\nu \le C\|G\|_{L^2}^2 \le C\delta^{-1}\|G\|_{L^2}^2,$$

whenever $\delta < 1$, which is our standing assumption.

It remains to retrace the main steps in the proof of (68), in the context of the estimate for $\int_N^\infty \|\partial_x^{-1}\mathcal{T}_{k_0(\nu)}\partial_x G\|_{L^2}^2 \nu^2 d\nu$. In fact, using the concrete formulas for $\mathcal{T}_{k_0(\nu)}$ and using the fact that $\mathcal{Q}$ acts invariantly on $P_{k_0}(L^2)$, we have

$$\begin{aligned}\partial_x^{-1}\mathcal{T}_{k_0(\nu)}\partial_x &= \partial_x^{-1}(\mathcal{Q} + i\frac{\delta}{k_0})^{-1}\partial_x^{-1}P_{k_0}\partial_x + \partial_x^{-1}(\mathcal{Q} + i\frac{\delta}{k_0})^{-1} \\ &\times P_{k_0}VP_{\neq k_0}\mathcal{M}^{-1}\partial_x^{-1}P_{\neq k_0}\partial_x = \\ &= (\mathcal{Q} + i\frac{\delta}{k_0})^{-1}\partial_x^{-1}P_{k_0} + \partial_x^{-1}(\mathcal{Q} + i\frac{\delta}{k_0})^{-1}P_{k_0}VP_{\neq k_0}\mathcal{M}^{-1}P_{\neq k_0}.\end{aligned}$$

Note that its first piece, $(\mathcal{Q} + i\frac{\delta}{k_0})^{-1}\partial_x^{-1}P_{k_0}$ is identical with what needed to be controlled earlier and that has been done in the proof of (68). The second piece, which is given by $\partial_x^{-1}(\mathcal{Q} + i\frac{\delta}{k_0})^{-1}P_{k_0}VP_{\neq k_0}\mathcal{M}^{-1}P_{\neq k_0}$, inserted in the appropriate integral, can be estimated in exactly the same way as in the proof of the corresponding piece for $\mathcal{T}_{k_0}$ and the estimate follows in the same fashion. We omit further details.

CRediT authorship contribution statement

Harrison Gaebler: Development of this project, Preparation of the manuscript and its revision. **Milena Stanislavova:** Development of this project, Preparation of the manuscript and its revision.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Milena Stanislavova Harrison Gaebler

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