DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement Volume 2005

Website: http://AIMsciences.org

pp. **824–832**

# ON THE GLOBAL ATTRACTOR FOR THE DAMPED BENJAMIN-BONA-MAHONY EQUATION

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**Abstract.** We present a new necessary and sufficient condition to verify the asymptotic compactness of an evolution equation defined in an unbounded domain, which involves the Littlewood-Paley projection operators. We then use this condition to prove the existence of an attractor for the damped Benjamin-Bona-Mahony equation in the phase space $H^1(\mathbf{R})$ by showing the solutions are point dissipative and asymptotically compact. Moreover the attractor is in fact smoother and it belongs to $H^{3/2-\varepsilon}$ for every $\varepsilon > 0$.

## 1. **Introduction and Preliminaries.**

In the literature on water waves there are numerous models describing the two-way propagation of long waves of small amplitude. Examples of such equations in $\mathbf{R^1}$ in which the competing effects of the nonlinearity and dispersion are of the same small order are $u_t + u_x + uu_x + u_{xxx} - \nu u_{xx} = 0$ (KdV-Burgers equation) and $u_t + u_x + uu_x - u_{xxt} - \nu u_{xx} = 0$ (regularized long wave or Benjamin-Bona-Mahony equation). More generally, consider the following Benjamin-Bona-Mahony (BBM) equation:

$$u_t - \Delta u_t - \nu \Delta u + div(f(u)) = g, \tag{1}$$

where $\nu$ is a positive constant. The BBM equation was proposed in [9] as a model for propagation of long waves which incorporates nonlinear dispersive and dissipative effects. The existence and uniqueness of solutions, as well as the decay rates of solutions for this equation were studied by many authors, see, for example, [6, 7, 9, 3, 4, 10]. When the equation is defined in a bounded domain, there exists finite dimensional global attractor, see [13, 25, 26]. Note that when the domain of the equation is unbounded there are additional difficulties when proving the existence of attractors because, in this case, the Sobolev embeddings are not compact. There are several methods which can be used to show the existence of attractors in standard Sobolev spaces when the equations are defined in unbounded domains. One can use energy equation technique to show that the weak asymptotic compactness is equivalent to the strong asymptotic compactness or decompose the solution operator into a compact part and an asymptotically small part. A third method is to prove that the solutions are uniformly small for large space and time variables by a cut-off function or by a weight function.

Our main goal in this paper is to present a new necessary and sufficient condition for an evolution equation to be asymptotically compact in a Sobolev space defined on an unbounded domain. As an application of this method we will investigate

2000 *Mathematics Subject Classification.* Primary: 35B40; Secondary: 35B41.
This material is based upon work supported by the NSF under grant # EPS-0236913 and matching support from the State of Kansas through Kansas Technology Enterprise Corporation.

the asymptotic behavior of the solutions of the following damped Benjamin-Bona-Mahony (BBM) equation defined on $\mathbf{R^1}$:

$$u_t - u_{txx} - \nu u_{xx} + \nu u + (u^2)_x = g(x), \tag{2}$$

where $\nu$ is a positive constant, $g \in L^2(\mathbf{R})$ and $(t,x) \in \mathbf{R^1_+} \times \mathbf{R^1}$.

**Remark:** In [22] we use this technique to prove the existence of local attractors for the Benjamin-Bona-Mahony equation (1) on $\mathbf{R}^3$, under the assumption that $\|g\|_{L^2} << 1$. That is there exists an $\varepsilon > 0$ and a ball $B_{H^1}(R)$ of radius $R = R(\varepsilon)$ such that whenever the sequence of initial conditions $\{u_n^0\} \in B_{H^1}(R)$ and $t_n \to +\infty$ $\{S(t_n)u_n^0\}$ is asymptotically compact.

The paper is structured as follows. In section 1 we give an inroduction and some preliminary results about attractors for evolution equations and methods from harmonic analysis. In section 2 we formulate and prove the Riesz-Rellich criteria for asymptotic compactness of evolution semigroups. In section 3 we establish the global well-posedness for the equation (2), we show that the orbits are point dissipative and use Riesz-Rellich criteria to prove their asymptotic compactness. This establishes the existence of attractor for the damped BBM equation on $\mathbf{R}$. We prove the following result.

**Theorem 1.** *For the damped Benjamin-Bona-Mahony equation* (2) *there exists a global attractor in* $H^1(\mathbf{R})$. *Moreover the attractor is in fact smoother and it belongs to* $H^{3/2-\varepsilon}$ *for every* $\varepsilon > 0$.

In what follows, we will need some basic facts from harmonic analysis, which we recall next. Denote by $\mathcal{S}$ the Schwartz class. Then given $f \in \mathcal{S}$, the Fourier transform of $f$ is defined by

$$\hat{f}(\xi) = \int\limits_{\mathbf{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

Fix an even function $\varphi$ in $C_0^\infty(\mathbf{R}^n)$, so that the support of $\varphi$ is in the annulus $1/2 \le |\xi| \le 2$ and $\sum\limits_{k=-\infty}^{\infty} \varphi(2^{-k}x) = 1 \qquad \text{for all} \qquad x \neq 0$. Define the operator $P_\delta$ via $\widehat{P_\delta f}(\xi) = \varphi(\delta^{-1}\xi)\hat{f}(\xi)$. Observe that $P_\delta$ essentially restricts the Fourier support of the function $f$ to the annulus $\delta/2 \le |\xi| \le 2\delta$ and $\sum\limits_{\delta \quad dyadic} P_\delta = 1$. Sometimes we will denote $P_\delta u$ simply by $u_\delta$ and call this the **Littlewood-Paley projection operator** at frequency $\delta$. We will also make use of the operators $P_{<\delta} := \sum_{\delta' < \delta} P_{\delta'}$, etc. Note that the kernel form of such operators is given by $P_\delta f = \delta^n \widehat{\varphi}(\delta \cdot) * f$ and thus, since $\|\delta^n \widehat{\varphi}(\delta \cdot)\|_{L^1} \lesssim 1$, we have that $P_\delta : L^p \to L^p$ for all $1 \le p \le \infty$. Here and after, by $f \lesssim g$, we mean that there is an absolute constant $C$ such that $f \le Cg$. By $f \sim g$, we mean $f \lesssim g$ and $g \lesssim f$. Throughout this paper, we denote by $\| \cdot \|_{L^p(\mathbf{R}^n)}$ the norm of $L^p(\mathbf{R}^n)$ and $\| \cdot \|$ the norm of $L^2(\mathbf{R}^n)$.

Before continuing, let us recall some basic definitions and results about attractors for evolution equations. Consider an initial value problem for an evolution equation

$$\frac{d}{dt} u(t) = F(u(t)), \quad u(0) = u_0,$$

defined on a Hilbert space $H$. This problem is well-posed if there exists for every $u_0 \in H$ a unique solution $u \in C([0, +\infty), H)$ and in this case we can define the solution semigroup $\{S(t)\}_{t\geq 0}$ by $S(t)u_0 = u(t)$. $S(t)$ maps $H$ into $H$, satisfies the semigroup properties

$$S(t+s) = S(t)S(s), S(0) = Id$$

and is continuous in the initial data for each $t \geq 0$.

$S(t)$ is **point dissipative** if there is a bounded set $B$ such that for any $u_0 \in H, S(t)u_0 \in B$ for all sufficiently large $t \geq 0$. $S(t)$ is **asymptotically compact** in $H$ if $S(t_n)u_n$ has a convergent subsequence for any bounded sequence $u_n$ when $t_n \to +\infty$.

$\mathcal{A} \subset H$ is called a **global attractor** for the evolution equation if it is **compact, invariant** ($S(t)\mathcal{A} = \mathcal{A},\ t \geq 0$) and **attracts every bounded set** $X$ ( $S(t)X \to \mathcal{A},\ t \to \infty$).

It is well known that to show the existence of an attractor one has to prove that $S(t)_{t\geq 0}$ is point dissipative and asymptotically compact. When working in infinite domain the second of these two steps is usually more difficult. We discuss next a sufficient condition for a dynamical system (evolution semigroup) to be asymptotically compact in a Sobolev space defined on an unbounded domain. This condition says roughly that for asymptotic compactness on unbounded domain it is necessary for the solutions to be uniformly small when time and space Fourier variables are large enough.

2. **Proof of Riesz-Rellich Criteria for Asymptotic Compactness.** Recall the following theorem due to Riesz (see Theorem XIII.66, p. 248, [21]).

**Proposition 1.** *Let $S \subseteq L^p(\mathbf{R}^n)$ with $1 \leq p < \infty$. Then $S$ is precompact in $L^p(\mathbf{R}^n)$ if and only if the following conditions are satisfied:*

*(1) $S$ is bounded in $L^p(\mathbf{R}^n)$;*

*(2) $f \to 0$ in $L^p$ sense at infinity uniformly in $S$, i.e., for any $\epsilon$, there is a bounded set $K \subset \mathbf{R}^n$ so that for all $f \in S$: $\int_{\mathbf{R}^n\setminus K} |f(x)|^p dx \leq \epsilon^p$;*

*(3) $f(\cdot - y) \to f$ uniformly in $S$ as $y \to 0$, i.e., for any $\epsilon$, there is $\delta$ so that $f \in S$ and $|y| < \delta$ imply that $\int_{\mathbf{R}^n} |f(x-y) - f(x)|^p dx \leq \epsilon^p$.*

We note that in a bounded domain $D$, the condition for compactness of a sequence in $L^p(D)$ reduces to (1) and (3), since (2) is obviously satisfied.

In the special and important case $p = 2$ instead of checking the hard to verify condition (3) in Proposition 1, one may equivalently check that the $L^2$ mass of the Fourier transform on the complements of large balls tends to zero. Our next proposition shows that the uniform vanishing of the Fourier transform implies equicontinuity and is in the spirit of Theorem XIII.65 in [21].

**Proposition 2.** *Assume $\{f_n\}_{n=1}^{\infty}$ is bounded in $L^2(\mathbf{R}^n)$. Then*

$$\sup_n \int_{|\xi|>M} |\hat{f}_n|^2 d\xi \to 0 \text{ as } M \to \infty$$

*implies that* $\sup_n \|f_n(.+h) - f_n(.)\|_{L^2} \to 0$ as $h \to 0$.

*Proof.* Take $\varepsilon > 0$ and select $M = M(\varepsilon)$ so that $\sup\limits_n \int\limits_{|\xi|>M} |\widehat{f_n}(\xi)|^2 d\xi \leq \varepsilon^2$.

Choose $\delta = \varepsilon/(M \sup\limits_n \|f_n\|_{L^2})$. For every $|h| \leq \delta$, we have by the Plancherel's theorem

$$\begin{aligned}
&\sup_n \|f_n(\cdot - h) - f_n(\cdot)\|_{L^2}^2 \leq \int |e^{2\pi i h\xi} - 1|^2 |\widehat{f_n}(\xi)|^2 d\xi \lesssim \\
\lesssim\ & \int_{|\xi|>M} |\widehat{f_n}(\xi)|^2 d\xi + h^2 \int_{|\xi|<M} |\xi|^2 |\widehat{f_n}(\xi)|^2 d\xi \lesssim \\
\lesssim\ & \varepsilon^2 + h^2 M^2 \sup_n \|f_n\|_{L^2}^2 \leq 2\varepsilon^2.
\end{aligned}$$

□

Consider a dynamical system $\{S(t)\}_{t\geq 0}$ and let $S(t)f_n$ solves an evolution equation with initial condition $f_n$. Suppose that $\sup_n \|f_n\|_{H^1} < B$. Let $t_n \to \infty$ and denote $u_n(t_n, .) = S(t_n)f_n$. We have the following criteria for asymptotic compactness.

**Proposition 3.** *Assume that*

- $\sup_n \|u_n(t_n, .)\|_{H^1} \leq C(B)$
- $\limsup_n \|u_n(t_n, .)\|_{H^1(|x|>N)} \to 0$ *as* $N \to \infty$
- $\limsup_n \|P_{>N} u_n(t_n, .)\|_{H^1} \to 0$ *as* $N \to \infty$

*Then the sequence* $\{u_n(t_n, .)\}$ *is precompact.*

Note that above, we have changed the sup to lim sup. This is possible, since the notion of precompactness is stable under taking subsequences.
We will apply this result to study the attractor for the damped Benjamin-Bona-Mahony equation .

3. **Existence of Attractor for the Damped BBM Equation.** In order to define the dynamical system for the equation (2) we need to prove global well-posedness for this equation in the space $H^1(\mathbf{R})$. First we need to consider the Helmholtz operator $(1-\partial_x^2)^{-1}$ defined as the inverse of the second order differential operator $(1-\partial_x^2)$. Alternatively, one defines it for (sufficiently) smooth functions $f$ via $\mathcal{F}((1-\partial_x^2)^{-1}f)(\xi) := (1+4\pi^2|\xi|^2)^{-1}\widehat{f}(\xi)$. The following simple lemma is a variant of an endpoint Sobolev embedding result, which will be needed in our estimates later on.

**Lemma 1.** *Let* $u, v \in L^2(\mathbf{R})$*. Then* $\left\|\partial_x(1-\partial_x^2)^{-1}(uv)\right\|_{L^2(\mathbf{R})} \lesssim \|u\|_2 \|v\|_2$.

*Proof.* By Plancherel's formula and Hölder's inequality, we have

$$|\mathcal{F}(\partial_x(1-\partial_x^2)^{-1}(uv))(\xi)| \lesssim <\xi>^{-1} \left| \int \widehat{u}(\xi-\eta)\widehat{v}(\eta) d\eta \right| \lesssim <\xi>^{-1} \|u\|_2 \|v\|_2,$$

where $<\xi> := (1+|\xi|^2)^{1/2}$. The result follows since $<\xi>^{-1} \in L^2_\xi(\mathbf{R})$, the weighted $L^2$ space. □

**Remark** One can obviously improve the result above to $\left\| |\partial_x|^s (1-\partial_x^2)^{-1}(uv)\right\|_{L^2(\mathbf{R})} \lesssim \|u\|_2 \|v\|_2$ for all $0 \leq s < 3/2$, where $\mathcal{F}(|\partial_x|^s f)(\xi) := |\xi|^s \widehat{f}(\xi)$.

Next, we rewrite the damped Benjamin-Bona-Mahony equation in evolution form using the operator $(1-\partial_x^2)^{-1}$.

$$\left\{ \begin{array}{l} u_t + \nu u + \partial_x(1-\partial_x^2)^{-1}(u^2) = (1-\partial_x^2)^{-1} g(x) \\ u(0,x) = f(x). \end{array} \right. \tag{3}$$

Note that the classical Benjamin-Bona-Mahony equation (1) looks very similar when written in evolution form.

$$\left\{ \begin{array}{l} u_t - \nu\Delta(1-\Delta)^{-1}u + div(1-\Delta)^{-1}(u + \frac{1}{2}u^2) = (1-\Delta)^{-1}g. \\ u(0) = f(x). \end{array} \right. \tag{4}$$

The only difference between (3) and (4) is that the term $\nu\Delta(1-\Delta)^{-1}u$ is replaced with $\nu u$. For large frequencies both terms are equlivalent for obviuos reasons, they only differ in low frequencies. It is exactly the accumulation of mass in low frequencies which is the main obstacle to showing the existence of attractors in dimension one for the classical Benjamin-Bona-Mahony equation . In fact in [2] there is a counterexample that shows that attractor does not exist in dimension one for the following Benjamin-Bona-Mahony equation $u_t - u_{xxt} - u_{xx} + u^2 u_x = 0$.

Next, we proceed to show local well-posedness for (3). Consider the equivalent integral equation:

$$u(t,x) = f(x) + \int_0^t \left(\nu u(s) - \partial_x(1-\partial_x^2)^{-1}(u^2) + (1-\partial_x^2)^{-1}g(x)\right) ds.$$

Denote by $F(u) = \nu u - \partial_x(1-\partial_x^2)^{-1}(u^2) + (1-\partial_x^2)^{-1}g$ the nonlinearity. It is easy to show that $F$ maps $H^1(\mathbf{R})$ into itself and is locally Lipschitz continuous. More precisely, there exists a constant $C$ such that

$$\|F(u)\|_{H^1} \le C\|u\|_{H^1}^2 + \nu\|u\|_{H^1} + C\|g\|_{L^2}, \tag{5}$$

$$\|F(u) - F(v)\|_{H^1} \le C\|u-v\|_{H^1}(1 + \max(\|u\|_{H^1}, \|v\|_{H^1})). \tag{6}$$

Then by the fixed point argument, we find that the equation is locally well-posed in $H^1(\mathbf{R})$. To prove global well-posedness we need an apriori estimate for the $\|u\|_{H^1}$. To do this we multiply the equation by $u$ and integrate in $x$ to get

$$\partial_t \int (u^2 + u_x^2)dx + \nu \int (u^2 + u_x^2)dx = \int g(x)udx.$$

Denote by $I(t) = \int (u^2 + u_x^2)dx$ and estimate that

$$\partial_t I + \nu I \le \|g\|_{L^2}\|u\|_{L^2} \le \|g\|_{L^2}\sqrt{I(t)}.$$

Multiply both sides by $e^{\nu t}$ and integrate from 0 to $T$ to get that

$$I(T)e^{\nu t} - I(0) \le \sup_{0\le t\le T} \|g\|_{L^2} e^{\nu t} \sup_{0\le t\le T} \sqrt{I(T)}.$$

If $J(T) = \sup\limits_{0\le t\le T} I(t)$, then we have that

$$J(T) \le J(0)e^{-\nu T} + \|g\|_{L^2}\sqrt{J(T)}.$$

Using Gronwall-type argument we see that

$$J(T) \le c(J(0)e^{-\nu T} + \|g\|_{L^2})$$

and thus $\|u\|_{H^1}$ is uniformly bounded. Thus the equation (3) is globally well-posed when $\|g\|_{\mathbf{H^1}} < \infty$ and $\|f\|_{\mathbf{H^1}} < \infty$. Moreover, the point dissipativeness follows from the estimate

$$\lim_{t\to+\infty} \|u\|_{\mathbf{H^1}} \le C\|g\|_{L^2}.$$

Next, we proceed to check the three conditions of the Riesz-Rellich criteria, namely the uniform boundedness of the orbits, the uniform smallness on the complements of large balls and the uniform continuity in $\mathbf{H^1}$. Uniform boundedness of the orbits is clear from the previous argument. To show that $\limsup_n \|u_n(t_n,.)\|_{H^1(|x|>N)} \to 0$ as $N \to \infty$ we consider a modified energy functional

$$E_{N,n} = \int (u_n^2 + (\partial_x u_n)^2)(1-\varphi(\frac{x}{N}))dx \ ,$$

where the smooth cutoff function $\varphi \in C^\infty(\mathbf{R})$ is defined as 1 if $|x| < 1$ and 0 if $|x| > 2$. We compute

$$E'_{N,n}(t) = 2\int (u_n(u_n)_t + (u_n)_x(u_n)_{xt})(1-\varphi(\frac{x}{N}))dx =$$

$$= 2\int u_n \left[-\nu u_n - \partial_x(1-\partial_x^2)^{-1}(u_n^2) + (1-\partial_x^2)^{-1} g\right](1-\varphi(\frac{x}{N})dx +$$

$$+2\int (u_n)_x \left[-\nu (u_n)_x - \partial_x^2(1-\partial_x^2)^{-1}(u_n^2) + \partial_x(1-\partial_x^2)^{-1} g\right](1-\varphi(\frac{x}{N})dx$$

Combining terms, integrating by parts and using the identity

$$\partial_x^2(1-\partial_x^2)^{-1} = (\partial_x^2 - 1)(1-\partial_x^2)^{-1} + (1-\partial_x^2)^{-1}$$

we get that

$$\partial_t E_{N,n} + 2\nu E_{N,n} = \frac{2}{N}\int u_n(1-\partial_x^2)^{-1}(u_n^2)\varphi'(\frac{x}{N})dx + \frac{2}{3N}\int u_n^3 \varphi'(\frac{x}{N})dx +$$

$$+\frac{2}{N}\int u_n \ \partial_x(1-\partial_x^2)^{-1} g \ \varphi'(\frac{x}{N})dx + 2\int u_n \ g(1-\varphi(\frac{x}{N}))dx$$

Next, we estimate the terms on the right hand side.

$$|\frac{2}{N}\int u_n(1-\partial_x^2)^{-1}(u_n^2)\varphi'(\frac{x}{N})dx| \le \frac{2}{N}\|u_n\|_{L^2}\left\|(1-\partial_x^2)^{-1}(u_n^2)\right\|_{L^2}$$

$$\le \quad \frac{2}{N}\|u_n\|_{L^2}\left\|u_n^2\right\|_{L^1} \le 2\frac{\|u_n\|_{L^2}^3}{N} \le \frac{c}{N}.$$

Last equality is due to the fact that the operator $(1-\partial_x^2)^{-1}$ maps $L^1(\mathbf{R^1})$ into $L^2(\mathbf{R^1})$. Also

$$|\frac{2}{3N}\int u_n^3\varphi'(\frac{x}{N})dx| \le \frac{2}{3N}\|u_n\|_{L^3}^3 \le \frac{2}{3N}\|u_n\|_{\mathbf{H^1}}^3 \le \frac{c}{N}.$$

Using the boundedness of the operator $\partial_x(1-\partial_x^2)^{-1}$ in $L^2$ we estimate

$$|\frac{2}{N}\int u \ \partial_x(1-\partial_x^2)^{-1} g \ \varphi'(\frac{x}{N})dx| \le \frac{c}{N}\|u_n\|_{L^2}\left\|\partial_x(1-\partial_x^2)^{-1}g\right\|_{L^2(|x|>N/2)}$$

$$\le \quad \frac{c}{N}\|u_n\|_{L^2}\|g\|_{L^2(|x|>N/2)} \le \frac{c}{N}\|g\|_{L^2(|x|>N/2)}.$$

Finally

$$|\int u_n \ g(1-\varphi(\frac{x}{N}))dx| \le \|u_n\|_{L^2}\|g\|_{L^2(|x|>N/2)} \le C\|g\|_{L^2(|x|>N/2)}$$

Altogether we have

$$\partial_t E_{N,n} + 2\nu E_{N,n} \le \frac{c}{N} + C\|g\|_{L^2(|x|>N/2)}.$$

Using again Gronwall type arguments we see that

$$\lim_{N\to\infty}\limsup_n \|u_n(t_n,.)\|_{H^1(|x|>N)} = \lim_{N\to\infty}\limsup_n E_{N,n}(t) \le$$

$$\le \quad \lim_{N\to\infty}\limsup_n e^{-\nu t_n}E_{N,n}(0) + \lim_{N\to\infty}\|g\|_{L^2(|x|>N/2)} = 0.$$

The last condition that we have to check is the uniform continuity in $\mathbf{H^1}$, i.e.

$$\sup_n \|P_{>N}\ u_n(t_n, .)\|_{\mathbf{H^1}} \to 0 \quad as\ N \to \infty.$$

For large $N$ we have that $\int |P_{>N}\ u_x|^2 dx >> \int |P_{>N}\ u|^2 dx$. Thus, suffices to show that

$$J^n_{>N}(t) = \int |P_{>N}(u_n)_x|^2 dx \to 0 \quad as$$

satisfies $\lim_{N\to\infty} \limsup_n J^n_{>N}(t_n) = 0$.
Again consider

$$\partial_t J^n_{>N}(t) = 2\int P_{>N}\ (u_n)_x\ P_{>N}(u_n)_{xt} dx =$$

$$= 2\int P_{>N}(u_n)_x\ P_{>N}(-\nu_n (u_n)_x - \partial_x^2(1-\partial_x^2)^{-1}(u_n^2) - \partial_x(1-\partial_x^2)^{-1} g) dx$$

We obtain that

$$\partial_t J^n_{>N}(t) + 2\nu\ J^n_{>N} =$$

$$= 2\int P_{>N}(u_n)_x\ P_{>N}(\partial_x^2(1-\partial_x^2)^{-1}(u_n^2)) dx + 2\int P_{>N}\ (u_n)_x\ P_{>N}(\partial_x(1-\partial_x^2)^{-1} g) dx$$

For the second term on the right we have the easy estimate

$$\begin{aligned} & \int P_{>N}(u_n)_x\ P_{>N}(\partial_x(1-\partial_x^2)^{-1} g) dx \\ \leq\ & \|P_{>N}(u_n)_x\|_{L^2} \left\|P_{>N}(\partial_x(1-\partial_x^2)^{-1} g\right\|_{L^2} \\ \leq\ & \sqrt{J^n_{>N}(t)}\frac{1}{N}\|g\|_{L^2} \leq \frac{c}{N}. \end{aligned}$$

To estimate the first term we will need the following lemma.

**Lemma 2.**

$$\left\|P_{>N}(u^2)\right\|_{L^2(\mathbf{R^1})} \leq \frac{c}{N}\|u\|^2_{\mathbf{H^1(R^1)}}$$

Using the lemma, we get

$$\int P_{>N}(u_n)_x\ P_{>N}(\partial_x^2(1-\partial_x^2)^{-1}(u_n^2)) dx \leq \|P_{>N}(u_n)_x\|_{L^2} \left\|P_{>N}\partial_x^2(1-\partial_x^2)^{-1}(u_n^2)\right\|_{L^2}$$

$$\leq \sqrt{J^n_{>N}(t)}\left\|P_{>N}(u_n^2)\right\|_{L^2} \leq \frac{c}{N}\sqrt{J^n_{>N}(t)}\|u_n\|^2_{\mathbf{H^1(R^1)}} \leq \frac{c}{N}.$$

Thus

$$\partial_t J^n_{>N}(t) + 2\nu\ J^n_{>N}(t) \leq \frac{c}{N}$$

and as before this means that

$$\lim_{N\to\infty} \limsup_n J^n_{>N}(t_n) = 0$$

.
Note that from this computation one gets that for every $u$ in the attractor, we have

$$\int |P_{>N} u_x(x)|^2 dx = \limsup_n \int |P_{>N}(u_n)_x(t_n, x)|^2 dx \leq C N^{-1}.$$

It follows that $u_x \in H^{1/2-\varepsilon}$, or $u \in H^{3/2-\varepsilon}$ for all $\varepsilon > 0$.
Next, we prove Lemma 2.

*Proof.* We will show that $\|P_{>N}(fg)\|_{L^2(\mathbf{R})} \leq \frac{C}{N}\|f\|_{H^1}\|g\|_{H^1}$. Furthermore, let $N \sim 2^s$. If we establish the estimate

$$\|(fg)_l\|_{L^2} \leq C2^{-l}\|f\|_{H^1}\|g\|_{H^1}, \tag{7}$$

the lemma would follow by dyadic summing (7) in $l \geq s-2$.
Note that since

$$\widehat{fg}(\eta) = \hat{f} * \hat{g}(\eta) = \int \hat{f}(\eta - \xi)\hat{g}(\xi)d\xi,$$

we have that $supp\ \widehat{fg} \subset supp\ \hat{f} + supp\ \hat{g}$.
Next,

$$P_l(fg) = \sum_{k,m} P_l(f_k g_m) = \sum_{k,m:|k-m|>3} P_l(f_k g_m) + \sum_{k,m:|k-m|\leq 3} P_l(f_k g_m)$$

However, if $|k-m| > 3$, it follows that $|\max(k,m) - l| \leq 3$. Indeed, if $|\max(k,m) - l| > 3$, we will have that $supp\hat{f_k} + supp\hat{g_m} \subset \{\xi : |\xi| \sim 2^{\max(k,m)}\}$, which is disjoint with $\{\xi : |\xi| \sim 2^l\}$, and therefore $P_l(f_k g_m) = 0$ for this configuration.

Similarly, if $|k-m| \leq 3$, it follows that $\min(k,m) > l-5$. Therefore,

$$P_l(fg) = \sum_{k,m:|k-m|>3\ \&|\max(k,m)-l|\leq 3} P_l(f_k g_m) + \sum_{k,m:|k-m|\leq 3\ \&\min(k,m)>l-5} P_l(f_k g_m).$$

Since the expressions are symmetric in $k, m$, let us assume without loss of generality that $k \geq m$. Estimate the first term by Hölder's inequality and Sobolev embedding, we obtain

$$\begin{aligned} & \sum_{k,m:|k-m|>3\ \&|k-l|\leq 3} \|f_k\|_{L^2}\|g_m\|_{L^\infty} \\ \leq\ & C\Big(\sum_{k:|k-l|\leq 3} 2^{-k}\|f\|_{H^1}\Big)\Big(\|g\|_{L^2} + \sum_{m>0} 2^{-m/2}\|g_m\|_{H^1}\Big) \\ \leq\ & C2^{-l}\|f\|_{H^1}\|g\|_{H^1}. \end{aligned}$$

The second term is estimated by

$$\begin{aligned} & \sum_{k,m:|k-m|\leq 3\ \&m>l-5} \|P_l(f_k g_m)\|_{L^2} \\ \leq\ & C \sum_{k,m:|k-m|\leq 3\ \&m>l-5} 2^{l/2}\|P_l(f_k g_m)\|_{L^1} \\ \leq\ & C \sum_{k,m:|k-m|\leq 3\ \&m>l-5} 2^{l/2}\|f_k\|_{L^2}\|g_m\|_{L^2} \\ \leq\ & C \sum_{k,m:|k-m|\leq 3\ \&m>l-5} 2^{l/2-m-k}\|f\|_{H^1}\|g\|_{H^1} \leq \\ \leq\ & C2^{-3l/2}\|f\|_{H^1}\|g\|_{H^1}. \end{aligned}$$

□

## REFERENCES

[1] F. Abergel, *Existence and finite dimensionality of the global attractor for evolution equations on unbounded domains*, J. Differential Equations,**83** (1990), 85–108.

[2] N. Akroune, *Comportement asymptotique de certaines equations faiblement amorties*,(2000), Ph.D. theses.

[3] J. Albert, *Dispersion of low-energy waves for the generalized Benjamin-Bona-Mahony equation*, J. Differential Equations, **63** (1986), 117-134.

[4] J. Albert, *On the decay of solutions of the generalized Benjamin-Bona-Mahony equation*, J. Math. Anal. Appl., **141**(1989), 527-537.

[5] C.J. Amick, J.L. Bona, M.E. Schonbek, *Decay of solutions of some nonlinear wave equations*, J. Differential Equations, **81**(1989), 1-49.

[6] J. Avrin, *The generalized Benjamin-Bona-Mahony equation in* $\mathbf{R}^n$ *with singular initial data*, Nonlinear Anal., **11** (1987), 139-147.

[7] J. Avrin, J.A. Goldstein, *Global existence for the Benjamin-Bona-Mahony equation in arbitrary dimensions*, Nonlinear Anal., **9** (1985), 861-865.

[8] A.V. Babin and M.I. Vishik, Attractors of Evolution Equations, North Holland, Amsterdam, 1991.

[9] T.B. Benjamin, J.L. Bona, J.J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. Londan , **272**(1972), 47-78.

[10] P. Biler, *Long time behaviour of solutions of the generalized Benjamin-Bona-Mahony equation in two-space dimensions*, Differential Integral Equations, **5**(1992), 891-901.

[11] J.L. Bona, P.J. Bryant, *A mathematical model for long waves generated by wavemakers in nonlinear dispersive systems*, Proc. Cambridge Philos. Soc., **73**(1973), 391-405.

[12] J.L. Bona, V.A. Dougalis, *An initial - and boundary-value problem for a model equation for propagation of long waves*, J. Math. Anal. Appl., **75**(1980), 503-522.

[13] A.O. Celebi, V.K. Kalantarov, M. Polat, *Attractors for the generalized Benjamin-Bona-Mahony equation, J. Differential Equations*, **157** (1999), 439-451.

[14] J.A. Goldstein, B.J. Wichnoski, *On the Benjamin-Bona-Mahony equation in higher dimensions*, Nonlinear Anal., **4** (1980), 665-675.

[15] O. Goubet, *Asymptotic smoothing effect for a weakly damped nonlinear Schrodinger equation in* $T^2$, J. Differential Equations, **165** (2000), 96–122.

[16] O. Goubet, *Asymptotic smoothing effect for weakly damped forced Korteweg-de Vries equations*, Discrete Contin. Dynam. Systems, **6** (2000), 625–644.

[17] O. Goubet, *Regularity of the attractor for a weakly damped nonlinear Schrödinger equation in* $\mathbf{R}^2$, Adv. Differential Equations, **3** (1998), 337–360.

[18] O. Goubet, *Regularity of the attractor for the weakly damped nonlinear Schrödinger equation*, Appl. Anal., **60** (1996), 99-119.

[19] O. Goubet, R. Rosa, *Asymptotic smoothing and the global attractor of a weakly damped KdV equation on the real line*, J. Differential Equations, **185** (2002), 25-53.

[20] I. Moise, R. Rosa, X. Wang, *Attractors for non-compact semigroups via energy equations*, Nonlinearity, **11** (1998), 1369-1393.

[21] M. Reed, B. Simon, Methods of Modern Mathematical Physics IV, Analysis of Operator, 1978

[22] M. Stanislavova, A. Stefanov, B. Wang, *Asymptotic Smoothing and attractors for the generalized Benjamin-Bona-Mahony equation on* $\mathbf{R}^3$, preprint.

[23] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1988.

[24] B. Wang, *Attractors for reaction diffusion equations in unbounded domains*, Physica D, **128** (1999), 41-52.

[25] B. Wang, *Strong Attracrors for the Benjamin-Bona-Mahony equation*, Appl. Math. Lett., **10** (1997), 23-28.

[26] B. Wang, W. Yang, *Finite dimensional behaviour for the Benjamin-Bona-Mahony equation*, J. Phys. A, Math. Gen., **30** (1997), 4877-4885.

Received September, 2004; revised April, 2005.

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