On Global Finite Energy Solutions of the Camassa-Holm Equation

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2 authors:

Milena Stanislavova
University of Kansas

Atanas Stefanov
University of Kansas

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On Global Finite Energy Solutions of the Camassa-Holm Equation

Milena Stanislavova and Atanas Stefanov

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ABSTRACT. *We consider the Camassa-Holm equation with data in the energy norm $H^1(\mathbf{R}^1)$. Global solutions are constructed by the small viscosity method for the frequency localized equations. The solutions are classical, unique and energy conservative. For finite band data, we show that global solutions for CH exist, satisfy the equation pointwise in time and satisfy the energy conservation law. We show that blow-up for higher Sobolev norms generally occurs in finite time and it might be of power type even for data in $H^{3/2-}$.*

1. Introduction and Statement of Results

In water wave theory, one usually takes asymptotical expansion in small parameter around a simple wave given by the Euler equation. Depending on the type of approximation that one chooses, several different models have been obtained. Probably the best known is the Korteweg- deVries equation

$$u_t + uu_x + u_{xxx} = 0 \ ,$$

where u stands for the fluid velocity. This is probably the most studied partial differential equation, because of the infinite number of preserved quantities and the applicability of the inverse scattering method.

In this article, we shall address the question of global existence and uniqueness for the Camassa-Holm equation

$$(CH) \quad \left| \begin{array}{l} u_t - u_{txx} = 2u_x u_{xx} + uu_{xxx} - 3uu_x \\ u(x,0) = u_0 \ . \end{array} \right. \tag{1.1}$$

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Camassa and Holm [3] derived (1.1) as an appropriate non-linear model for water wave motion in shallow channels. In the same article and later in collaboration with Hyman [2], they have studied some basic properties of the solutions (including an explicit formulas for special solutions) and also the typical non-linear phenomenon of peakon-antipeakon interaction. Related equations and hierarchies of equations were earlier considered by Fuchssteiner and Focas [13], without an explicit reference to this explicit model. Theirs was a concentrated effort to produce models for soliton interactions with biHamiltonian structure (and thus infinite number of conserved quantities).

We also note that in the work of H.-H. Dai on thin compressible elastic rods (see also [4]), the equation

$$(CH_\gamma) \qquad u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx})$$

has been proposed to model the small-amplitude (axial-radial) deformation waves. Here $u(x,t)$ measures the radial stretch relative to an equilibrium. It turns out that the physical parameter γ ranges from -29.5 to 3.41.

The Camassa-Holm equation was object of intensive investigation since its appearence. We will not even try to give a comprehensive overview of the known results (a seemingly impossible task). Instead, we will only focus on results that are relevant to our goal — the existence and uniqueness of the solutions. For the biHamiltonian structure and the related question of symmetries we refer the interested reader to [12].

The Camassa-Holm equation enjoys infinite number of conserved quantities. Most of them, however are nonlocal in nature, thus making them ineffective for the standard energy estimates approach. In fact, there are only three known local preserved quantities: Preservation of mass yields $\int u\,dx = \text{const}$ and the Hamiltonians

$$\begin{aligned} I &= \int_{\mathbf{R}^1} \left(u^2 + u_x^2\right) dx = \text{const} \;, \\ K &= \int_{\mathbf{R}^1} \left(u^3 + uu_x^2\right) dx = \text{const} \;. \end{aligned}$$

Constantin and Strauss exploited the preservation of the two Hamiltonians (together with the existence results in [15], or the consequent article [6]) to obtain orbital stability of the traveling wave solutions up to the existence time. That presents one more motivation as to why one is interested in whether the Hamiltonians I and K are preserved for H^1 solutions. For the existence theory, we start with the local well-posedness result of Li and Olver, [15] in $H^{3/2+}$. Note that the exponent $3/2+$ matches the sharp local well-posedness result for hyperbolic equations in 1 D. Olver and Li have actually provided several blow-up scenarios showing that the blow-up cannot be prevented by taking small data in any sense. Concerning less smooth data, Constantin and Molinet, [6] have shown that if the data u_0 is in the Sobolev space H^1 [6] and if $y_0 = u_0 - \partial_x^2 u_0$ is a positive Radon measure, then the Equation (1.1) has an unique solution, which belongs to $C((0,\infty), H^1(\mathbf{R}^1))$, satisfy the appropriate conservation laws and $u - \partial_x^2 u$ is also a positive Radon measure.

We must also mention the work of Xin and Zhang, [20, 21]. They have shown the existence of weak solutions for general data $u_0 \in H^1$, [20]. Their argument yields solutions in a distributional sense. In [21], Xin and Zhang obtain uniqueness under certain additional hypothesis on the weak solutions, obtained in [20]. As a result, one does not get conservation of the H^1 energy.

In this article, we consider an alternative approach to the existence of global weak solutions. First, we consider the frequency localized variants, for which we prove global existence, uniqueness and conservation of the H^1 energy. These are models motivated by the spectral method for solving nonlinear equations in finite domains. Moreover, by the very structure of these equations, we obtain classical solutions corresponding to a general H^1 data. In the case of finite band initial data, we are able to then take a limit of such solutions to obtain solutions to the standard CH equation, which are energy conservative. Let us mention that these are solutions that satisfy the CH equation *pointwise in time* and in a stronger (L^2) sense in x. We are also able to show that the convergence $u(t,\cdot) \to u_0$ is in L^2 and almost everywhere sense, although we do not pursue these issues here. Before we state our results, we will need to give a precise definition of a solution.

Definition 1. We say that u is an solution of the Camassa-Holm equation in the time interval $[0,T)$ in strong (L^2) sense, if $u \in L^\infty([0,T), H^1_x(\mathbf{R}^1)) \cap C([0,T), L^2_x)$, $u_t \in L^\infty((0,T), L^2_x)$ and it satisfies $u(x,0) = u_0(x)$

$$u_t + \frac{1}{2}\partial_x\big(u^2\big) + \frac{1}{2}\partial_x\big(1-\partial_x^2\big)^{-1}\big(u_x^2\big) + \partial_x\big(1-\partial_x^2\big)^{-1}\big(u^2\big) = 0\,. \tag{1.2}$$

The equality in (1.2) is to be understood as L^2 functions.

The precise meaning of the operators $\big(1-\partial_x^2\big)^{-1}$, $\partial_x\big(1-\partial_x^2\big)^{-1}$ above as well as the relationship between (1.2) and the Camassa-Holm equation will be explained in the sequel. For convenience, let us denote $F(u,v) = \partial_x(u^2)/2 + \partial_x\big(1-\partial_x^2\big)^{-1}(v^2/2+u^2)$, so that the nonlinearity of (1.2) is $F(u,u_x)$.

In this work, we will be concerned with the frequency localized analogue of the Camassa-Holm equation. We will need to define the frequency localization operator $P_{<N}$. Namely, let $P_{<N}$ be an operator acting by $\widehat{P_{<N}f}(\xi) = \eta(2^{-N}\xi)\hat{f}(\xi)$ for some smooth cutoff η (see Section 2 below).

Theorem 1. *The frequency localized Camassa-Holm equation*

$$\left|\begin{array}{l} u_t + P_{<N}F(P_{<N}u, P_{<N}u_x) = 0 \\ u(x,0) = u_0 \end{array}\right.$$

has global solution in H^1, provided $u_0 \in H^1(\mathbf{R}^1)$. The solution is unique in the class $u_t \in L^\infty_{t,\mathrm{loc}}L^2_x$, $u \in L^\infty H^1$. Moreover, it satisfies the energy conservation law

$$\int_{\mathbf{R}^1}\big(u^2(T,\cdot) + u_x^2(T,\cdot)\big)\,dx = \int\big(u_0^2 + (\partial_x u_0)^2\big)\,dx\,.$$

Our motivation for considering such models is the spectral method approach to that problem in finite intervals. As is usual in numerical schemes, consider the CH equation $u_t + F(u,u_x) = 0$ in a finite space interval $-M \le x \le M$. One then replaces the solution u by a *finite trigonometric polynomial*, and one replaces the nonlinearity by a finite trigonometric polynomial as well. These approximations are performed by applying the Fourier projections $\tilde{P}_{\le N}$ on the periodic functions, that is $\tilde{P}_{\le N}(\sum_j a_j e^{2\partial[iijx/M}) = \sum_{j=-N}^{N} a_j e^{2\pi ijx/M}$. These frequency localized approximations have the following important features:

- They satisfy the energy conservation law.
- When solving on appropriate finite interval $-M \leq x \leq M$, one has a representation of the solution u as a finite (degree N) trigonometric polynomial, whose coefficients $\{a_n\}$ will evolve with time and satisfy a corresponding system of *ordinary differential equations.*
- $\tilde{P}_{\leq N} u \to u$ as $N \to \infty$.

Unfortunately, taking limits of the solutions of Theorem 1 as $N \to \infty$ proves out to be hard. To the best of our knowledge, it is an open question whether global (weak) solutions exist *and preserves energy* in the context of the original Equation (1.2), given arbitrary data $u_0 \in H^1$. We note that one cannot expect in general to have a global solution, which has smoothness higher than H^1. In fact, Constantin and Escher have shown that smooth solutions blow up if and only if $\liminf_{t\to T} \inf_x u_x(x,t) = -\infty$ and this actually occurs for very specific choices of (very smooth) initial data. We have the following result.

Theorem 2 (Existence for finite band data). *Let the initial data $u_0 \in H^1(\mathbf{R}^1)$ is finite band, that is $P_{>M} u_0 = 0$ for some M. Then there exists a L^2 solution u in the sense of Definition 1. Moreover, the Hamiltonian is preserved*

$$\int_{\mathbf{R}^1} u^2 + (\partial_x u)^2 \, dx = \int_{\mathbf{R}^1} u_0^2 + (\partial_x u_0)^2 \, dx \ ,$$

and for every $T > 0$, $\lim_{M\to\infty} \sup_{0\leq t\leq T} \|P_{>M} u(t,\cdot)\|_{H^1} = 0$.

Remarks.

- The solution u is obtained as an H^1 limit of some subsequence of the solutions u^N in Theorem 1.
- The set of finite band initial data is a *dense set* in H^1. In particular, according to the necessary and sufficient conditions of McKean [16], wave breaking (i.e., $\liminf_{t\to T} \inf_x u_x(x,t) = -\infty$) occurs for some finite band initial data and finite time T. Nevertheless, according to Theorem 2 the (weak) solution can be extended beyond T. This is in contrast with the results of Constantin and Molinet, [6], who consider data u_0 with $u_0 - \partial_x^2 u_0 \in M(R)_+$ for which wave breaking never occurs.
- The property $\lim_{M\to\infty} \|P_{>M} u(t,\cdot)\|_{H^1} = 0$, is trivially satisfied for any given t. The statement here is nontrivial because one is taking supremum over all times in a given interval $(0,T)$, i.e., $\lim_{M\to\infty} \sup\limits_{0<t<T} \|P_{>M} u(t,\cdot)\|_{H^1} = 0$.

Our next theorem shows that any norm of the solution in the scale H^s may blow up for $s > 1$.

Theorem 3. *Let $1 < s < 3/2$. Then there exists solution of the Camassa-Holm equation whose H^s-norm blows up in finite time. More specifically, one can find an initial data in the space $H^{3/2-}$, for which there exists time $t_0 > 0$ such that the corresponding solution $\|u(\cdot,t)\|_{H^s}$ blows up at time t_0 and $\|u(\cdot,t)\|_{H^s} \geq |t - t_0|^{-(s-1)}$. For the same initial data, one has that there exists a time t_0 and $a < b$, so that*

$$\int_0^{t_0} \int_a^b |\partial_x u(t,x)|^3 \, dx \, dt = \infty \ ,$$

showing that the a priori estimate for the solutions (Proposition 3.2, [20])

$$\int_0^T \int_a^b |\partial_x u(t,x)|^{3-\varepsilon}\, dx\, dt \le C(T,\varepsilon)\ ,$$

is the best possible.

2. Background Results and Notations

2.1 Some Basic Results of Phase Space Analysis

Define the Fourier transform for functions in the Schwartz class as

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int f(x) e^{-2\pi i \langle \xi, x\rangle}\, dx\ .$$

One then can extend to L^2 functions by approximating L^2 functions with functions in the Schwartz class. One also has the Fourier inversion formula

$$\mathcal{F}^{-1}(f)(x) = \check{f}(x) = \int \widehat{f}(\xi) e^{2\pi i \langle \xi, x\rangle}\, d\xi\ .$$

We will also define the frequency restriction operators

$$\begin{aligned} \widehat{Q_0 f}(\xi) &= \chi_{[-1,1]}(\xi)\hat{f}(\xi) \\ \widehat{Q_{<N} f}(\xi) &= \chi_{[-N,N]}(\xi)\hat{f}(\xi) \quad \text{for} \quad N>0\ . \end{aligned}$$

Define an even function $\eta \in C_0^\infty(\mathbf{R}^1)$ so that $\eta(\xi) = 1$ for all $|\xi| < 1$ and $\operatorname{supp}\eta \subset (-2,2)$. Associated with η are the smooth Littlewood-Paley projection operators

$$\begin{aligned} \widehat{P_1 f}(\xi) &= \eta(\xi)\hat{f}(\xi) \\ \widehat{P_{<N} f}(\xi) &= \eta(\xi/N)\hat{f}(\xi) \quad \text{for} \quad k>0\ . \end{aligned}$$

Observe that since η is even, one has $P_{<N} f = N\hat{\eta}(N\cdot) * f$, where *the kernel is real valued.* Note also that $|\partial_x^\alpha N\hat{\eta}(Nx)| \le C_\alpha |x|^{-\alpha-1}$, where C_α is independent of N.

Introduce $P_{>k} := \mathrm{Id} - P_{<k}$ and $P_n = P_{<n+1} - P_{<n}$. Next, define the inhomogeneous Sobolev spaces H^s (for any real number s) by

$$\|f\|_{H^s\left(\mathbf{R}^n\right)} = \left(\int_{\mathbf{R}^n} \left|\widehat{f}(\xi)\right|^2 < \xi >^{2s}\ d\xi \right)^{1/2}\ ,$$

where $< \xi > := (1+|\xi|^2)^{1/2}$. Consider the Helmholtz operator $\left(1-\partial_x^2\right)^{-1}$ defined as the inverse of the second order differential operator $(1-\partial_x^2)$. Alternatively, one defines it for (sufficiently) smooth functions f via

$$\mathcal{F}\left(\left(1-\partial_x^2\right)^{-1} f\right)(\xi) := \left(1+4\pi^2|\xi|^2\right)^{-1}\widehat{f}(\xi)\ .$$

The following lemma is a variant of an endpoint Sobolev embedding result, which will be needed in our estimates later on.

Lemma 1. *Let $u, v \in L^2(\mathbf{R}^1)$. Then*

$$\left\| \partial_x \left(1-\partial_x^2\right)^{-1}(uv) \right\|_{L^2(\mathbf{R}^1)} \lesssim \|u\|_2 \|v\|_2 \,.$$

Proof. By Plancherel's formula and Hölder's inequality, we have

$$\left|\mathcal{F}\left(\partial_x\left(1-\partial_x^2\right)^{-1}(uv)\right)(\xi)\right| \lesssim <\xi>^{-1} \left| \int \widehat{u}(\xi-\eta)\widehat{v}(\eta)\, d\eta \right| \lesssim <\xi>^{-1} \|u\|_2 \|v\|_2 \,.$$

The result follows since $<\xi>^{-1} \in L^2_\xi(\mathbf{R}^1)$. □

Remark. One can obviously improve the result to

$$\left\| |\partial_x|^s \left(1-\partial_x^2\right)^{-1}(uv) \right\|_{L^2(\mathbf{R}^1)} \lesssim \|u\|_2 \|v\|_2 \,,$$

for all $0 \le s < 3/2$, where $\mathcal{F}(|\partial_x|^s f)(\xi) := |\xi|^s \widehat{f}(\xi)$. In the sequel, we shall need the Calderón commutator estimate, which roughly states that the commutator of a Calderón-Zygmund operator with the multiplication operator is a smoothing operator of order one, that is its argument *gains* one derivative as an L^2 function.

Lemma 2. *Suppose $T : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is a convolution type operator, $Tf = K * f$, whose kernel K satisfies $|\partial_x^\alpha K(x)| \le C_\alpha |x|^{-n-\alpha}$ away from the origin, for every multiindex α. Then for every Lipschitz function a, one has the following estimates for the commutator $[T, M_a]f = T(M_a f) - M_a(Tf)$:*

$$\|\partial_x [T, M_a] f\|_{L^2\left(\mathbf{R}^n\right)} \le C \left\|a'\right\|_{L^\infty} \|f\|_{L^2\left(\mathbf{R}^n\right)} \,,$$

where the constant C is independent of f and a.

2.2 Camassa-Holm Equation in a Conservation Law Form

We recast (CH) using the Helmholtz operator in a form better suited for our purposes. Indeed, an elementary computation shows that one can rewrite it (at least formally) as

$$u_t - u_{txx} = 2u_x u_{xx} + u u_{xxx} - 3uu_x = \frac{1}{2}\partial_x^3\left(u^2\right) - \frac{1}{2}\partial_x\left(u_x^2\right) - \frac{3}{2}\partial_x\left(u^2\right) ,$$

and therefore by applying $\left(1-\partial_x^2\right)^{-1}$ to both sides,

$$\left| \begin{array}{l} u_t + \frac{1}{2}\partial_x\left(u^2\right) + \frac{1}{2}\partial_x\left(1-\partial_x^2\right)^{-1}\left(u_x^2\right) + \partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2\right) = 0 \\ u(x,0) = u_0(x) \,. \end{array} \right. \tag{2.1}$$

Note that in (2.1) one encounters at most one spatial derivative on the solution, which will be essential in our analysis of weak solutions. In some instances, one can use the following equivalent reformulation of (2.1). Observe that the "peakon" $\varphi(x) = e^{-|x|}/2$ is the fundamental solution of the operator $(1-\partial_x^2)$, i.e.,

$$\left(1-\partial_x^2\right)^{-1} f(x) = e^{-|\cdot|}/2 * f = \frac{1}{2}\int_{\mathbf{R}^1} e^{-|x-y|} f(y)\, dy \,.$$

One can also write the operator $\partial_x\left(1-\partial_x^2\right)^{-1}$ as

$$\partial_x\left(1-\partial_x^2\right)^{-1} f(x) = -\frac{1}{2}\int_{\mathbf{R}^1} \operatorname{sgn}(x-y)e^{-|x-y|} f(y)\,dy$$

and therefore (2.1) becomes

$$u_t + \frac{1}{2}\left(u^2 + \varphi * \left(u^2 + \frac{u_x^2}{2}\right)\right)_x = 0$$

as in [5].

3. Small Viscosity Approximation

In this section, we consider the "small" viscosity approximation of the Camassa-Holm equation, namely

$$\left|\begin{array}{l} u_t + \frac{1}{2}\partial_x\left(u^2\right) + \frac{1}{2}\partial_x\left(1-\partial_x^2\right)^{-1}\left(u_x^2\right) + \partial_x\left(1-\partial_x^2\right)^{-1}\left(u^2\right) = \varepsilon\partial_x^2 u \\ u(x,0) = u_0(x)\ . \end{array}\right. \tag{3.1}$$

We will show that for every $\varepsilon > 0$, (3.1) has an unique solution, which dissipates energy.

Proposition 1. *For every $\varepsilon > 0$, the Equation* (3.1) *is uniquely and globally solvable given $u_0 \in H^1(\mathbf{R}^1)$ and*

$$I(t) = \int_{\mathbf{R}^1} u^2 + u_x^2\,dx \le \int_{\mathbf{R}^1} u_0^2 + (\partial_x u_0)^2\,dx\ .$$

The result relies on the classical theory for nonlinear heat equations together with the energy conservation for the Camassa-Holm equation. Recall that for the equation

$$\left|\begin{array}{l} u_t - \Delta u = F(u, \nabla u) \\ u(x,0) = u_0(x) \end{array}\right. \tag{3.2}$$

there is the following classical local existence result (cf. [17], p. 316).

Lemma 3. *Suppose that the nonlinearity F in* (3.2) *satisfies*

$$\left\{\begin{array}{l} \|F(u,\nabla u) - F(v,\nabla v)\|_{H^s} \le M_R\|u-v\|_{H^{s+1}} \\ \text{whenever} \qquad \|u\|_{H^{s+1}},\ \|v\|_{H^{s+1}} \le R \end{array}\right. \tag{3.3}$$

for some $s \ge 0$. Then, there exists time $T > 0$ depending only on $\|u_0\|_{H^{s+1}}$, such that the parabolic Equation (3.2) *has an unique local solution*

$$u \in C\left([0,T], H^{s+1}\left(\mathbf{R}^n\right)\right) \bigcap C^1\left((0,T], H^s\left(\mathbf{R}^n\right)\right)\ .$$

Moreover, for every $T > t_0 > 0$ one has $u \in C^\infty(\mathbf{R}^n \times [t_0, T))$.

In fact, a close inspection of the proof shows that $T = T(\|u_0\|_{H^s})$ is a decreasing function of $\|u_0\|_{H^s}$ and $\lim_{h\to 0} T(h) = \infty$. In other words, "smaller" data is expected to produce solutions with longer lifespan.

The proof of Lemma 3 relies on standard smoothing properties of the heat semigroup $e^{t\Delta}$, namely

$$\begin{cases} \left\|e^{t\Delta} f\right\|_{H^{s+1}} \lesssim t^{-1/2}\|f\|_{H^s} \,, \\ \left\|\left(e^{t\Delta} - \mathrm{Id}\right) f\right\|_{H^s} \lesssim t^{1/2}\|f\|_{H^{s+1}} \,. \end{cases} \tag{3.4}$$

Hence, the equation

$$\left| \begin{array}{l} u_t - \varepsilon\Delta u = F(u, \nabla u) \\ u(x, 0) = u_0(x) \end{array} \right. \tag{3.5}$$

has a unique local solution with the prescribed smoothness properties in Lemma 3 and with a lifespan at least $T_\varepsilon = T_\varepsilon(\|u_0\|_{H^{s+1}}) \sim \varepsilon T(\|u_0\|_{H^{s+1}})$, whenever (3.3) is satisfied.

Proof of Proposition 1. Set $s = 0$. We verify (3.3) for the Camassa-Holm nonlinearity

$$F(u, u_x) = \frac{1}{2}\partial_x\left(u^2\right) + \frac{1}{2}\partial_x\left(1 - \partial_x^2\right)^{-1}\left(u_x^2\right) + \partial_x\left(1 - \partial_x^2\right)^{-1}\left(u^2\right) . \tag{3.6}$$

One clearly has

$$\begin{aligned} F(u, u_x) - F(v, v_x) &= \frac{1}{2}\partial_x[(u - v)(u + v)] + \frac{1}{2}\partial_x\left(1 - \partial_x^2\right)^{-1}[(u_x - v_x)(u_x + v_x)] \\ &\quad + \partial_x\left(1 - \partial_x^2\right)^{-1}[(u - v)(u + v)] . \end{aligned}$$

By Hölder's inequality and the Sobolev embedding $H^1(\mathbf{R}^1) \hookrightarrow L^\infty(\mathbf{R}^1)$,

$$\begin{aligned} \|\partial_x[(u - v)(u + v)]\|_{L^2(\mathbf{R}^1)} &\lesssim \|u - v\|_{H^1}\|u + v\|_{L^\infty} + \|u + v\|_{H^1}\|u - v\|_{L^\infty} \\ &\lesssim \|u - v\|_{H^1}\left(\|u\|_{H^1} + \|v\|_{H^1}\right) , \end{aligned}$$

which shows that (3.3) holds with $M_R \sim R$. The term $\partial_x\left(1 - \partial_x^2\right)^{-1}[(u - v)(u + v)]$ is handled in a similar manner (it is in fact easier due to the smoothing of $(1 - \partial_x^2)^{-1}$). For the term $\partial_x\left(1 - \partial_x^2\right)^{-1}[(u_x - v_x)(u_x + v_x)]$, we use Lemma 1. We have

$$\begin{aligned} \left\|\partial_x\left(1 - \partial_x^2\right)^{-1}[(u_x - v_x)(u_x + v_x)]\right\|_{L^2(\mathbf{R}^1)} &\lesssim \|(u_x - v_x)\|_{L^2}\|(u_x + v_x)\|_{L^2} \\ &\lesssim \|u - v\|_{H^1}\left(\|u\|_{H^1} + \|v\|_{H^1}\right) . \end{aligned}$$

This shows that (3.3) is satisfied and we obtain a solution of (3.1). By the smoothness of the solution for $0 < t < T_\varepsilon$, we have

$$I(t) = \int_{\mathbf{R}^1} \left(u^2 + u_x^2\right) dx \le I(0) . \tag{3.7}$$

For (3.7), one checks as in the preservation of the Hamiltonian for the Camassa-Holm equation (see Corollary 1 in the Appendix) that

$$I'(t) = 2\int (uu_t + u_x u_{xt})\, dx = 2\varepsilon\int (uu_{xx} + u_x u_{xxx})\, dx = -2\varepsilon\int \left(u_x^2 + u_{xx}^2\right) dx \le 0. \tag{3.8}$$

Thus, $I(t)$ is a decreasing function and $I(t) \le I(0)$.

We note that this step is justified due to the smoothness properties of the solution u and cannot be performed in general for the solution of (CH) unless we have some additional smoothness information. Based on (3.7), we can iterate our local existence result to a global one. Indeed, take $t_0 = T_\varepsilon/2$ and observe that by the energy dissipation $\|u(\cdot, t_0)\|_{H^1} \leq \|u_0\|_{H^1}$. We can run the same argument now starting from t_0. The lifespan of the solution starting at t_0 is going to be at least $T_\varepsilon\left(\|u(\cdot, t_0)\|_{H^1}\right) \geq T_\varepsilon = T_\varepsilon\left(\|u_0\|_{H^1}\right)$ and so on. By taking small, but fixed increments in time, one produces global solution in that fashion. Moreover, at each step, we have the dissipation law (3.7). □

4. Frequency Localized (CH) Equations

In this section, we consider the following "frequency localized" version of the Cammasa-Holm equation for fixed N:

$$\left| \begin{array}{l} u_t + P_{<N} F(P_{<N} u, P_{<N} u_x) = 0 \\ u(x, 0) = u_0 \end{array} \right. \tag{4.1}$$

and the corresponding "small" viscosity equation

$$\left| \begin{array}{l} u_t + P_{<N} F(P_{<N} u, P_{<N} u_x) = \varepsilon u_{xx} \\ u(x, 0) = u_0 \; . \end{array} \right. \tag{4.2}$$

We note that $\|P_{<N}\|_{L^2 \to L^2} = 1$ and $P_{<N}$ is a convolution type operator and therefore commutes with $|\partial_x|^s$ and ∂_x. For fixed $\varepsilon > 0$, one can use these properties to show (just as in the case of Cammasa-Holm equation) that the estimates on the non-linearities of Lemma 3 are satisfied and therefore there is a smooth solution in some small interval $(0, T_\varepsilon)$, which satisfies the equation in classical sense. Now, one can apply $Q_{>4N}$ on both sides of (4.2) and since $Q_{>(4N)} P_{<N} = 0$, one finds that $Q_{>4N} u^N_\varepsilon$ satisfies a *linear* parabolic equation with data $Q_{>4N} u_0$ and therefore

$$u^N_\varepsilon = Q_{<4N} u^N_\varepsilon + Q_{>4N} e^{\varepsilon t \partial_x^2} u_0 \; . \tag{4.3}$$

It is straightforward to check, that the solutions of (4.2) still enjoy the energy dissipation property (3.8). That enables us to iterate the local result to a global one just as in the previous section. In addition, observe that according to the energy dissipation

$$\frac{dI}{dt} = -2\varepsilon \int \left((u^\varepsilon_x)^2 + (u^\varepsilon_{xx})^2 \right) dx \; ,$$

we have (after integration in time)

$$2\varepsilon \int_0^T \int \left((u^\varepsilon_x)^2 + (u^\varepsilon_{xx})^2 \right) dx \leq -\int_0^T I'(t)\, dt \leq I(0) = \|u_0\|^2_{H^1} \; . \tag{4.4}$$

We show now that the viscosity limit as $\varepsilon \to 0$ makes sense. This is the essence of Theorem 1, whose proof we present here.

Proof of Theorem 1 (Existence). We construct the solution as a limit of the solutions to the viscosity approximation (4.2) u^ε. To that end, we have to show that u^ε is a compact sequence in H^1, whose limit will be shown to be a solution to (4.1).

We make use of the Riesz criterion for compactness in H^1, (e.g., [18], p. 247). Namely, we have to verify

$$\sup_{\varepsilon>0} \left\| u^\varepsilon \right\|_{H^1} < \infty \tag{4.5}$$

$$\sup_{\varepsilon>0} \left\| u^\varepsilon(\cdot + h) - u^\varepsilon(\cdot) \right\|_{H^1} \to 0 \qquad \text{as} \qquad h \to 0 \tag{4.6}$$

$$\sup_{\varepsilon>0} \left\| u^\varepsilon \right\|_{H^1(|x|>M)} \to 0 \qquad \text{as} \qquad M \to \infty\,. \tag{4.7}$$

Clearly, $\sup_{\varepsilon>0} \|u^\varepsilon\|_{H^1} \le \|u_0\|_{H^1}$ by energy dissipation, so (4.5) is verified. For (4.6), observe that by Lemma 6 from the Appendix, it is enough to show that

$$\sup_{\varepsilon>0} \left\| Q_{>M} u^\varepsilon \right\|_{H^1} \to 0 \qquad \text{as} \qquad M \to \infty\,. \tag{4.8}$$

To verify (4.8), use the representation (4.3). Take $M > 8N$. We get

$$\sup_{\varepsilon>0} \left\| Q_{>M} u^\varepsilon \right\|_{H^1} = \sup_{\varepsilon>0} \left\| Q_{>M} e^{\varepsilon t \partial_x^2} u_0 \right\|_{H^1} \le \| Q_{>M} u_0 \|_{H^1} \to 0 \qquad \text{as} \qquad M \to \infty\,.$$

For (4.7), we have the following.

Lemma 4. *Let $u_0 \in H^1(\mathbf{R}^1)$. Define the energy of the solution u_ε in $\{x : |x| > M\}$ by*

$$J(t) = \int_{\mathbf{R}^1} \left(u_\varepsilon^2 + u_{\varepsilon x}^2 \right)(1 - \chi(x/M))\, dx\,,$$

where χ is a smooth cutoff adapted to $(-1, 1)$. Then

$$\frac{d}{dt} J(t) \ \le\ C M^{-1} \|u_0\|_{H^1}^3 + C\varepsilon M^{-2} \|u_0\|_{H^1}^2\,. \tag{4.9}$$

We will present the proof of the lemma in the Appendix. Let us indicate how to finish the proof of (4.7) based on it. Indeed, integrating (4.9), we obtain

$$|J(T)| \le |J(0)| + CTM^{-1} \|u_0\|_{H^1}^3 + CT\varepsilon M^{-2} \|u_0\|_{H^1}^2$$

for every $T > 0$. Fix T. For every $\delta > 0$, we can find $M_1\left(T, \|u_0\|_{H^1}\right)$, so that $CTM^{-1}\|u_0\|_{H^1}^3 + CT\varepsilon M^{-2}\|u_0\|_{H^1}^2 \le \delta/2$ and $M_2(T, u_0)$ so that $|J(0)| \le \delta/2$. Thus, if $M > \max(M_1, M_2)$, then $\|u^\varepsilon\|_{H^1(|x|>M)} \le \delta$, which is (4.7). This shows that one can extract an H^1 convergent subsequence $\{u^{\varepsilon_n}\}$ from $\{u^\varepsilon\}_{\varepsilon>0}$. We verify now that the limit u solves the equation in L^2 sense. First, consider the term $\partial_x\left(1 - \partial_x^2\right)^{-1}[(u_x^\varepsilon)^2]$. One has by Lemma 1

$$\begin{aligned} &\left\| P_{<N} \partial_x \left(1 - \partial_x^2\right)^{-1} \left[\left(P_{<N} u_x^{\varepsilon_n}\right)^2 \right] - P_{<N} \partial_x \left(1 - \partial_x^2\right)^{-1} \left[\left(P_{<N} u_x\right)^2 \right] \right\|_{L^2} \\ &\quad \lesssim \left\| P_{<N} u_x^{\varepsilon_n} - P_{<N} u_x \right\|_{L^2} \left\| P_{<N} u_x^{\varepsilon_n} + P_{<N} u_x \right\|_{L^2} \le 2 \left\| u^{\varepsilon_n} - u \right\|_{H^1} \|u_0\|_{H^1}\,, \end{aligned}$$

clearly converging to zero. Similarly, one proves $\partial_x\left(1-\partial_x^2\right)^{-1}[(u^\varepsilon)^2] \to \partial_x\left(1-\partial_x^2\right)^{-1}[(u)^2]$ in L^2 sense. For $\partial_x(u^\varepsilon)^2$, one estimates by Sobolev embedding

$$\left\| \partial_x \left(u^\varepsilon\right)^2 - \partial_x u^2 \right\|_{L^2} \le \left\| u^\varepsilon - u \right\|_{H^1} \left\| u^\varepsilon + u \right\|_{L^\infty} \lesssim \left\| u^\varepsilon - u \right\|_{H^1} \|u_0\|_{H^1}\,.$$

This shows that for the nonlinearity of the frequency localized equation, we have shown

$$\sup_{0\le t\le T} \left\| P_{<N}F\left(P_{<N}u^{\varepsilon_n}, P_{<N}u_x^{\varepsilon_n}\right) - P_{<N}F(P_{<N}u, P_{<N}u_x)\right\|_{L^2} \to 0 \,. \tag{4.10}$$

Next, we establish that $u_t \in L_t^\infty L_x^2$ and $u_t + P_{<N}F(P_{<N}u, P_{<N}u_x) = 0$ in L_x^2 sense. Indeed, we have for the classical solutions u^{ε_n} by (4.4)

$$\left\|u_t^{\varepsilon_n} + P_{<N}F\left(P_{<N}u^{\varepsilon_n}, P_{<N}u_x^{\varepsilon_n}\right)\right\|_{L_x^2 L^2(0,T)} = \varepsilon_n \left\|u_{xx}^{\varepsilon_n}\right\|_{L^2 L^2(0,T)} \le \sqrt{\varepsilon_n}\|u_0\|_{H^1} \to 0 \,.$$

Combining the last conclusion with (4.10), we get that the sequence $\{u_t^{\varepsilon_n}\}$ is convergent in $L^2(0,T)L_x^2$ and for the limit $v : v + P_{<N}F(P_{<N}u, P_{<N}u_x) = 0$. By Lemma 1, it is easy to see that $\|v(t,\cdot)\|_{L^2} = \|P_{<N}F(P_{<N}u, P_{<N}u_x)\|_{L^2} \le C\|u_0\|_{H^1}^2$. On the other hand, for every smooth test function ψ with $\psi(0,\cdot) = 0$, we have

$$\langle v, \psi\rangle = \lim_n \left\langle u_t^{\varepsilon_n}, \psi\right\rangle = -\lim_n \left\langle u^{\varepsilon_n}, \psi_t\right\rangle = -\langle u, \psi_t\rangle \,.$$

It follows that u_t exists and $u_t = v \in L_t^\infty L_x^2$ and it satisfies (in L^2 sense)

$$u_t + P_{<N}F(P_{<N}u, P_{<N}u_x) = 0 \,.$$

Uniqueness and conservation law.

Let

$$I(t) = \int \left(u^2 + u_x^2\right) dx \,.$$

By the representation (4.3) for u_ε^N, we find that the limit u will satisfy $u = Q_{<4N}u + Q_{>4N}u_0$, whence $u_t = Q_{<4N}u_t$, since $Q_{>4N}u_0$ is time independent. Thus,

$$I(t) = \int \left((Q_{<4N}u)^2 + (Q_{<4N}u_x)^2\right) dx + \int \left((Q_{>4N}u_0)^2 + (\partial_x Q_{>4N}u_0))^2\right) dx \,,$$

where the second term is constant in time. Differentiation in time yields,

$$I'(t) \;=\; 2\int (Q_{<4N}u\, Q_{<4N}u_t + Q_{<4N}u_x\, Q_{<4N}u_{tx})\, dx \,. \tag{4.11}$$

This is justified, since both $Q_{<4N}u_t$ and $Q_{<4N}u_{tx} \in L^2(\mathbf{R}^1)$. Using the equation (which is satisfied in L^2 sense) and the fact that $Q_{<4N}^2 = Q_{<4N}$ and $Q_{<4N}P_{<N} = P_{<N}$, we rewrite (4.11) as

$$\begin{aligned}
I'(t) &= 2\,\langle Q_{<4N}u, Q_{<4N}u_t\rangle + 2\,\langle Q_{<4N}u_x, Q_{<4N}u_{tx}\rangle \\
&= 2\left\langle Q_{<4N}^2 u, u_t\right\rangle + 2\left\langle Q_{<4N}^2 u_x, u_{tx}\right\rangle \\
&= -2\left\langle Q_{<4N}^2 u, P_{<N}F(P_{<N}u, P_{<N}u_x)\right\rangle - 2\left\langle Q_{<4N}^2 u_x, \partial_x P_{<N}F(P_{<N}u, P_{<N}u_x)\right\rangle \\
&= -2\left\langle P_{<N}Q_{<4N}^2 u, F(P_{<N}u, P_{<N}u_x)\right\rangle - 2\left\langle P_{<N}Q_{<4N}^2 u_x, \partial_x F(P_{<N}u, P_{<N}u_x)\right\rangle \\
&= -2\,\langle P_{<N}u, F(P_{<N}u, P_{<N}u_x)\rangle - 2\,\langle P_{<N}u_x, \partial_x F(P_{<N}u, P_{<N}u_x)\rangle \,.
\end{aligned}$$

One now computes that $I'(t) = 0$ by (7.3) from Corollary 1 in the Appendix (this is simply the conservation law applied to $P_{<N}u$.) Therefore,

$$\int \left(u^2 + u_x^2\right) dx = \int \left(u_0^2 + (\partial_x u_0)^2\right) dx \,,$$

proving the energy conservation for the frequency localized equation.

For the uniqueness, assume that $\tilde{u}$ and $\tilde{v}$ are two solutions to (4.1) with the same initial data u_0. Observe next, that the difference will satisfy

$$\left| \begin{array}{l} (\tilde{u}-\tilde{v})_t + P_{<N}(F(P_{<N}u, P_{<N}u_x) - F(P_{<N}v, P_{<N}v_x)) = 0 \\ (\tilde{u}-\tilde{v})(x,0) = 0\,. \end{array} \right.$$

Note that since $Q_{>4N}P_{<N} = 0$, we have $Q_{>4N}(\tilde{u}-\tilde{v}) = 0$. In particular, $\tilde{u}-\tilde{v}$ is a smooth function, which belongs to L^2 along with all of its derivatives. Consider the energy of the difference equation,

$$D(t) = \int (\tilde{u}-\tilde{v})^2 + (\tilde{u}_x - \tilde{v}_x)^2\, dx\,.$$

By assumption $(\tilde{u}-\tilde{v})_t \in L^\infty_{t,\mathrm{loc}}L^2$ and in addition $(\tilde{u}-\tilde{v})_{tx} = \partial_x Q_{<4N}(\tilde{u}-\tilde{v})_t \in L^\infty_{t,\mathrm{loc}}L^2$. Differentiate with respect to t to get

$$D'(t) = 2\int (\tilde{u}-\tilde{v})(\tilde{u}-\tilde{v})_t + (\tilde{u}_x - \tilde{v}_x)(\tilde{u}_x - \tilde{v}_x)_t\, dx\,.$$

Use the equation and introduce the functions $u = P_{<N}\tilde{u}$, $v = P_{<N}\tilde{v}$. We get

$$\begin{aligned} D(T) - D(0) = \int_0^T D'(t) = -\int_0^T \int & (u_x - v_x)\partial_x^2[(u-v)(u+v)] \\ & + (u_x - v_x)\partial_x^2(1-\partial_x^2)^{-1}[(u_x - v_x)(u_x+v_x)] \\ & + 2(u_x - v_x)\partial_x^2(1-\partial_x^2)^{-1}[(u-v)(u+v)]\, dx\,. \end{aligned}$$

Using the fact that both u and v are Fourier supported in $|\xi| \le 4N$ and $D(0) = 0$, we can (very crudely) estimate

$$D(T) \le CN^2 \int_0^T \|(u-v)(t,\cdot)\|_{H^1}^2 \left(\|u\|_{H^1} + \|v\|_{H^1}\right) dt \le CN^2 \int_0^T D(t)\, dt\,,$$

since $\max\left(\|u\|_{H^1}, \|v\|_{H^1}\right) < \max\left(\|\tilde{u}\|_{H^1}, \|\tilde{v}\|_{H^1}\right) < \infty$ by assumption. Thus, one gets by Gronwall's inequality

$$0 \le D(T) \le D(0)e^{CN^2 T}\,.$$

We conclude $D(T) = 0$ for all $T > 0$, since $D(0) = 0$ and the uniqueness follows. □

5. Existence of Global Finite Energy Solutions of the Camassa-Holm Equation

In this section, we show that the limit of solutions to the frequency localized CH equations converge to solutions of the original equation, provided the initial data is band-limited. For initial data $u_0 \in H^1$, take the sequence of the solutions to the frequency localized CH Equations (4.1) for $n = 1, 2 \ldots.$ This is uniformly bounded sequence in H^1, and in fact by the energy preservation (see Theorem 1) $\left\|u^N\right\|_{H^1} = \|u_0\|_{H^1}$. Thus, there is a weak (and pointwise) limit point u. Next, we show that for some subsequence N_k: $\left\|u^{N_k} - u\right\|_{H^s} \to 0$ for every $0 \le s < 1$ (but not necessarily for $s = 1$). By the Riesz-Rellich criterion for compactness in L^2, we have to show

- $\sup_N \left\|u^N\right\|_{H^s} < \infty$.
- $\sup_N \left\|u^N(\cdot + h) - u^N(\cdot)\right\|_{H^s} \to 0$ as $h \to 0$.
- $\sup_N \left\|u^N\right\|_{H^s(|x|>M)} \to 0$ as $M \to \infty$.

But $\sup_N \left\|u^N\right\|_{H^s} \le \sup_N \left\|u^N\right\|_{H^1} = \|u_0\|_{H^1} < \infty$. Similarly, we have already verified by means of Lemma 4, that

$$\sup_N \left\|u^N\right\|_{H^s(|x|>M)} \le \sup_N \left\|u^N\right\|_{H^1(|x|>M)} = O\left(M^{-1}\right) .$$

It remains to show $\sup_N \left\|u^N(\cdot + h) - u^N(\cdot)\right\|_{H^s} \to 0$. But, as we have already pointed out in our previous discussion, it suffices to verify

$$\sup_N \int_{|\xi|\ge R} \left(1 + |\xi|^{2s}\right)\left|\widehat{u}^N(\xi)\right|^2 d\xi \to 0 \qquad \text{as} \qquad R \to \infty .$$

This follows from the control of the H^1 norm, since

$$\sup_N \int_{|\xi|\ge R} \left(1 + |\xi|^{2s}\right)\left|\widehat{u}^N(\xi)\right|^2 d\xi \le R^{2(s-1)} \sup_N \left\|u^N\right\|_{H^1}^2 = R^{2(s-1)}\|u_0\|_{H^1}^2 \to_R 0 .$$

Thus, we are guaranteed a convergent subsequence in H^s (and thus in L^∞ as well).

Fix time $T : 0 < T < \infty$. By the Riesz-Rellich compactness criteria, in order to show that the convergence (of a subsequence) is in H^1 it remains to show that for every $\delta > 0$, there exists M_δ, so that for all $M > M_\delta$, $\sup_{0\le t\le T} \sup_N \left\|P_{\ge M} u^N\right\|_{H^1} \le \delta$. Assume the contrary, that is the existence of some $\delta_0 > 0$, so that $\sup_{0\le t\le T} \sup_N \left\|P_{\ge M} u^N\right\|_{H^1} \ge \delta_0$ for all M. This is easily seen to imply that there exists M_0, so that for all $M > M_0$, one has

$$\sup_{0\le t\le T} \sup_N \left\|P_{\ge M/10} u^N\right\|_{H^1} \le 2 \sup_{0\le t\le T} \sup_N \left\|P_{\ge(10M)} u^N\right\|_{H^1} . \tag{5.1}$$

Indeed, otherwise, one constructs an increasing sequence M_n so that $\sup_N \left\|P_{\ge M_n} u^N\right\|_{H^1} \ge 2 \sup_N \left\|P_{\ge M_{n+1}} u^N\right\|_{H^1}$, which leads to a contradiction, since

$$\|u_0\|_{H^1} \ge \sup_{0\le t\le T} \sup_N \left\|P_{\ge M_0} u^N\right\|_{H^1} \ge 2^n \sup_{0\le t\le T} \sup_N \left\|P_{\ge M_n} u^N\right\|_{H^1} \ge 2^n \delta_0 \to \infty .$$

We will now proceed to give an estimate on the high-frequency portion of the energy, that is, we consider

$$I_{>M}^N(t) = \left\|P_{>M} u^N\right\|_{H^1}^2 = \int \left|P_{>M} u^N\right|^2 + \left|P_{>M} u_x^N\right|^2 dx .$$

By the equation,

$$\begin{aligned}
\partial_t I_{>M}^N &= 2 \int \left(P_{>M} u^N P_{>M} u_t^N + P_{>M} u_x^N P_{>M} u_{tx}^N\right) dx \\
&= 2 \int P_{>M} u^N P_{>M} P_{<N} F\left(P_{<N} u^N, P_{<N} u_x^N\right) dx \\
&\quad + 2 \int P_{>M} u_x^N P_{>M} P_{<N} \partial_x F\left(P_{<N} u^N, P_{<N} u_x^N\right) dx .
\end{aligned}$$

Note that the last calculation is justified, because the formal integral representation for $\partial_t I^N_{>M}$ contains only smooth L^2 entries (recall u_t is Fourier supported in $\{|\xi| \leq 2^{N+2}\}$). Conjugate by $P_{>M}P_{<N}$ and call $v = P_{<N}u^N$. We obtain

$$\partial_t I^N_{>M} = 2\int \left(P^2_{>M}v\right)F(v, v_x)\,dx + 2\int \left(P^2_{>M}v_x\right)\partial_x F(v, v_x)\,dx\ .$$

Denote $\tilde{P}_{>M} := P^2_{>M}$. Note that $\tilde{P}_{>M}$ has similar properties to $P_{>M}$. Write $v = \tilde{P}_{>M}v + \tilde{P}_{\leq M}v =: z + w$. In other words, we denote the low frequency portion of v by w and the high-frequency portion by z. We get

$$\begin{aligned}\partial_t I^N_{>M} &= 2\int zF(z, z_x)\,dx + 2\int z_x\partial_x F(z, z_x)\,dx\\ &+ 2\int z\left(\frac{1}{2}\partial_x\left(2zw + w^2\right) + \frac{1}{2}\partial_x\left(1-\partial_x^2\right)^{-1}\left(2z_xw_x + w_x^2 + 4zw + 2w^2\right)\right)dx\\ &+ 2\int z_x\left(\frac{1}{2}\partial_x^2\left(2zw + w^2\right) + \frac{1}{2}\partial_x^2\left(1-\partial_x^2\right)^{-1}\left(2z_xw_x + w_x^2 + 4zw + 2w^2\right)\right)dx\ .\end{aligned}$$

Note that

$$2\int zF(z, z_x)\,dx + 2\int z_x\partial_x F(z, z_x)\,dx = 0\ ,$$

since $z = \tilde{P}_{>M}P_{<N}u^N$ is smooth.[1] For the terms

$$\begin{aligned}&\int z\partial_x(zw) + z\partial_x\left(1-\partial_x^2\right)^{-1}(2z_xw_x + 4zw) + z_x\left(1-\partial_x^2\right)^{-1}(2z_xw_x + 4zw)\,dx\\ &\quad + 2\int z_x(-z_xw_x - 4zw)\,dx\ ,\end{aligned}$$

we use Hölder's inequality and Lemma 1 to estimate by $C\|z\|^2_{H^1}(\|w\|_{L^\infty} + \|w_x\|_{L^\infty}) \lesssim M^{1/2}\|z\|^2_{H^1}\|w\|_{H^1}$, since the Fourier support of $w \subset \{|\xi| \leq M\}$.

Next, we make the observation that the really dangerous term $\int z_xz_{xx}w\,dx$ arises, but is transformed into $-\frac{1}{2}\int z_x^2w_x\,dx$ via integration by parts.

With the last formula in mind (and for some terms the fact that $\left(1-\partial_x^2\right)^{-1}$ is bounded on L^2), all expressions appearing above that are *quadratic* in the z variable, are estimated by placing z and z_x in L^2 and the remaining w, w_x or w_{xx} in L^∞. We obtain the following estimate for these terms: $C\|z\|^2_{H^1}(\|w\|_{L^\infty} + \|w_x\|_{L^\infty} + \|w_{xx}\|_{L^\infty}) \lesssim M^{3/2}\|z\|^2_{H^1}\|w\|_{H^1}$.

It remains to consider the terms that are *linear* in z and *quadratic* in w. These are

$$\begin{aligned}H &= \frac{1}{2}\int z\partial_x\left(w^2\right) + z\partial_x\left(1-\partial_x^2\right)^{-1}\left(w_x^2 + 2w^2\right)dx\\ &\quad + \frac{1}{2}\int z_x\partial_x^2\left(w^2\right) + z_x\partial_x^2\left(1-\partial_x^2\right)^{-1}\left(w_x^2 + 2w^2\right)dx\ .\end{aligned}$$

[1]This is an identity satisfied for all smooth functions, regardless of whether they are solutions to CH or not.

Since the Fourier support of z is inside $\{|\xi| > M/2\}$, it follows from the Plancherel's formula that

$$\int z\partial_x(Q_{\le M/4}w)^2 + z\partial_x\big(1-\partial_x^2\big)^{-1}(Q_{\le M/4}w_x)^2 + 2(Q_{\le M/4}w)^2\big)\,dx$$
$$+\int z_x\partial_x^2(Q_{\le M/4}w)^2 + z_x\partial_x^2\big(1-\partial_x^2\big)^{-1}(Q_{\le M/4}w_x)^2 + 2(Q_{\le M/4}w)^2\big)\,dx = 0\,.$$

Thus, one can rewrite H as

$$\begin{aligned}2H = &\int z\partial_x\big(2Q_{>M/4}wQ_{\le M/4}w + (Q_{>M/4}w)^2\big)\,dx\\
&+\int z\partial_x\big(1-\partial_x^2\big)^{-1}\big(2Q_{>M/4}w_xQ_{\le M/4}w_x + (Q_{>M/4}w_x)^2\big)\,dx\\
&+\int z\big(4Q_{>M/4}wQ_{\le M/4}w + 2(Q_{>M/4}w)^2\big)\,dx\\
&+\int z_x\partial_x^2\big(2Q_{>M/4}wQ_{\le M/4}w + (Q_{>M/4}w)^2\big)\,dx\\
&+\int z_x\partial_x^2\big(1-\partial_x^2\big)^{-1}\big(2Q_{>M/4}w_xQ_{\le M/4}w_x + (Q_{>M/4}w_x)^2\big)\,dx\\
&+2\int z_x\partial_x^2\big(1-\partial_x^2\big)^{-1}\big(2Q_{>M/4}wQ_{\le M/4}w + (Q_{>M/4}w)^2\big)\,dx\,.\end{aligned}$$

Observe that each term above contains a z or z_x and additionally either $Q_{>M/4}w$ or $Q_{>M/4}w_x$ or $Q_{>M/4}w_{xx}$. As always, we estimate via the Hölder's inequality by placing the terms like z, z_x and $Q_{>M/4}w$, $Q_{>M/4}w_x$, and $Q_{>M/4}w_{xx}$ in L^2 and the remaining term in L^∞. Note that for some terms, we need to estimate $\|Q_{<M/4}w_x\|_{L^\infty} \lesssim \int_{|\xi|\le M/4}|\widehat{w_x}(\xi)\,d\xi| \lesssim M^{1/2}\|\widehat{w_x}\|_{L^2} \sim M^{1/2}\|w\|_{H^1}$. We get

$$H \le CM^{3/2}\|z\|_{H^1}\|Q_{>M/4}w\|_{H^1}\|w\|_{H^1}\,.$$

By (5.1), one concludes that

$$\|Q_{>M/4}w\|_{H^1} = \left\|Q_{>M/4}\tilde P_{\le M}P_{\le N}u^N\right\|_{H^1} \le \left\|Q_{>M/4}u^N\right\|_{H^1} \le 2\sqrt{I^N_{>M}}\,.$$

Also, $\|z\|_{H^1} = \|\tilde P_{>M}P_{\le N}u^N\|_{H^1} \le \|\tilde P_{>M}u^N\|_{H^1} \le \sqrt{I^N_{>M}}$.

Collecting all the estimate for $\partial_t I^N_{>M}$ yields

$$\partial_t I^N_{>M}(t) \le CM^{3/2}\|u_0\|_{H^1}I^N_{>M}(t)\,,$$

or after integration in $(0,T)$

$$I^N_{>M}(T) - I^N_{>M}(0) \le CM^{3/2}\|u_0\|_{H^1}\int_0^T I^N_{>M}(t)\,dt\,. \tag{5.2}$$

Choose now M large enough so that $I^N_{>M}(0) = \int|P_{>M}u_0|^2 + |P_{>M}\partial_x u_0|^2\,dx = 0$. This is possible from the fact that u_0 is finite band. But $I^N_{>M}(t)$ is a smooth function of time, it vanishes at $t = 0$ and satisfies (5.2). By Gronwall's inequality, $I^N_{>M}(t) = 0$ for all

$0 \leq t \leq T$ in contradiction with $\sup_{0\leq t\leq T} \sup_N \left\| P_{\geq M} u^N \right\|_{H^1} \geq \delta_0$. Hence, we have shown the uniform (in N) vanishing of the Fourier transform and this by the Riesz-Relich criteria guarantees the existence of a subsequence of u^N converging to u in H^1. Clearly, u satisfies the CH equation in L^2 sense and preserves the H^1 energy for all time, since $\|u(t,\cdot)\|_{H^1} = \lim_k \left\|u^{N_k}(t,\cdot)\right\|_{H^1} = \|u_0\|_{H^1}$.

6. Blowing-Up of Solutions

In this section we will show that there are solutions of the Camassa-Holm equation (the "multipeakon" solutions), whose H^s-norms blow-up in finite time for any $s > 1$ even though the initial data is in the space $H^{3/2-\varepsilon}$. In this sense the global existence result from Section 1 is sharp, meaning that one can not obtain unconditional global well posedness in H^s, $s > 1$. The example that we give uses the peakon-antipeakon interaction, studied in [2] and subsequently by [1]. The multipeakon solution, constructed by Camassa and Holm has the form

$$u(x,t) = \frac{1}{2}\sum_{j=1}^{n} m_j(t)\exp(-2|x - x_j(t)|)$$

and represents n interacting traveling waves. Here x_j are the positions and m_j are the momenta.

Proof of Theorem 3. We will use the explicit formulas in [1], Section 7. For the derivative $u_x(x,t)$ of the multipeakon solution u on certain interval (x_k, x_{k+1}) it is shown in [1] that $u_x(x,t) = \frac{\alpha}{t-t_0} + O(1)$ as $t \to t_0$ with a constant $\alpha > 0$. The length of this interval is $x_{k+1} - x_k = \alpha_0(t-t_0)^2 + O((t-t_0)^3)$ for some constant $\alpha_0 > 0$. Thus,

$$\|u_x\|_{L^p} \geq \left(\int_{x_k}^{x_{k+1}} (u_x)^p\, dx\right)^{1/p} \geq \frac{C}{(t-t_0)^{(p-2)/p}}$$

for some constant $C > 0$. The Sobolev embedding theorem then implies that $\|u\|_{H^{1+\varepsilon}} \geq \|u_x\|_{L^p}$ for $\varepsilon = \frac{p-2}{p}$. Thus, $\|u\|_{H^{1+\varepsilon}} \geq \frac{C}{(t-t_0)^{\varepsilon}}$. We also see that

$$\int_0^{t_0}\int_{x_k}^{x_{k+1}} |u_x|^p \gtrsim \int_0^{t_0} \frac{1}{|t-t_0|^{p-2}}\, dt\ ,$$

which is divergent if $p \geq 3$. □

7. Appendix

7.1 Decay of the Localized Energy

The purpose of this section is to study the decay of the energy functional $J(t)$. We will in fact prove a much more general lemma than Lemma 4 for solutions of the viscosity approximations (3.1) and (4.2). We note that the lemma remains true for the actual solutions of the Camassa-Holm equation (that is for $\varepsilon = 0$), as long as they are smooth enough to justify the manipulations below (say C^2).

Lemma 5. *Let u be the unique smooth solution to either* (3.1)*, or* (4.2)*. Define the energy functionals*

$$J(t) := \int \left(u^2 + u_x^2\right)(1 - \chi(x/M))\, dx$$

and

$$S(t) := \int \left(u^2 + u_x^2\right)\chi(x/M)\, dx$$

where $0 \leq \chi \leq 1$ *is a smooth cutoff function adapted to* $(-1, 1)$*. Then*

$$\left|J'(t)\right| \leq CM^{-1}\|u_0\|_{H^1}^3 + CM^{-2}\varepsilon\|u_0\|_{H^1}^2 \,, \tag{7.1}$$

$$\left|S'(t)\right| \leq CM^{-1}\|u_0\|_{H^1}^3 + CM^{-2}\varepsilon\|u_0\|_{H^1}^2 \tag{7.2}$$

for some absolute constant C.

As a corollary, take $M \to \infty$ to recover the energy conservation law for smooth solutions of (1.2).

Corollary 1. *Let u be a* C^2 *smooth solution of* (1.2) *up to time T. Then*

$$\int uu_t + u_x u_{tx}\, dx = 0\,.$$

In particular, the energy

$$I(t) = \int \left(u^2 + u_x^2\right) dx$$

is conserved up to time T. For the frequency localized version, if u is a solution of (3.1)*, then*

$$\int P_{<N}uP_{<N}u_t + P_{<N}u_x P_{<N}u_{tx}\, dx = 0 \tag{7.3}$$

and in particular $I(t)$ *is preserved.*

Proof of Lemma 5. We first work in the case of (3.1) and then we indicate how to modify the proof, to accommodate solutions of the frequency localized version (4.2). Next, we only consider the energy functional J (which is the only thing needed in the preceding sections), the proof for S is similar. Since u is known to be smooth and satisfies (3.1) classically, we differentiate $J(t)$ and use the equation. We get

$$\begin{aligned}
J'(t) &= 2\int (uu_t + u_x u_{xt})(1 - \chi(x/M))\, dx = -\int u\partial_x\left(u^2\right)(1 - \chi(x/M))\, dx \\
&- \int u\partial_x\left(1 - \partial_x^2\right)^{-1}\left(u_x^2\right)(1 - \chi(x/M))\, dx - 2\int u\partial_x\left(1 - \partial_x^2\right)^{-1}\left(u^2\right)\big)(1 - \chi(x/M)) \\
&- \int u_x\partial_x^2\left(u^2\right)(1 - \chi(x/M))\, dx - \int u_x\partial_x^2\left(1 - \partial_x^2\right)^{-1}\left(u_x^2\right)(1 - \chi(x/M))\, dx \\
&- 2\int u_x\partial_x^2\left(1 - \partial_x^2\right)^{-1}\left(u^2\right)\big)(1 - \chi(x/M))\, dx + \varepsilon\int u\partial_x^2 u(1 - \chi(x/M))\, dx \\
&+ \varepsilon\int u_x\partial_x^3 u(1 - \chi(x/M))\, dx\,.
\end{aligned}$$

Each term is handled separately by integration by parts, or by using the identity $\partial_x^2(1-\partial_x^2)^{-1} = -\operatorname{Id} + (1-\partial_x^2)^{-1}$ or both. Note that the boundary terms disappear by the decay properties of the solution. We have

$$
\begin{aligned}
&-\int u\partial_x(u^2)(1-\chi(x/M))\,dx = -\frac{2}{3M}\int u^3\chi'(x/M)\,dx\ ,\\
&-\int u\partial_x(1-\partial_x^2)^{-1}(u_x^2)(1-\chi(x/M))\,dx = \int u_x(1-\partial_x^2)^{-1}(u_x^2)(1-\chi(x/M))\,dx\\
&-\frac{1}{M}\int u(1-\partial_x^2)^{-1}(u_x^2)\chi'(x/M)\,dx\ ,\\
&-2\int u\partial_x(1-\partial_x^2)^{-1}(u^2))(1-\chi(x/M))\,dx = 2\int u_x(1-\partial_x^2)^{-1}(u^2))(1-\chi(x/M))\,dx\\
&-\frac{2}{M}\int u(1-\partial_x^2)^{-1}(u^2)\chi'(x/M)\,dx\ ,\\
&-\int u_x\partial_x^2(u^2)(1-\chi(x/M))\,dx = -\int u_x^3(1-\chi(x/M))\,dx\\
&-\frac{1}{M}\int u_x^2u\chi'(x/M)\,dx\ ,\\
&-\int u_x\partial_x^2(1-\partial_x^2)^{-1}(u_x^2)(1-\chi(x/M))\,dx = \int u_x^3(1-\chi(x/M))\,dx\\
&-\int u_x(1-\partial_x^2)^{-1}(u_x^2)(1-\chi(x/M))\,dx\ ,\\
&-2\int u_x\partial_x^2(1-\partial_x^2)^{-1}(u^2))(1-\chi(x/M))\,dx = \frac{2}{3M}\int u^3\chi'(x/M)\,dx\\
&-2\int u_x(1-\partial_x^2)^{-1}(u^2)(1-\chi(x/M))\,dx\ .
\end{aligned}
$$

Finally, the two extra terms coming from the small viscosity approximation are

$$
\begin{aligned}
&\varepsilon\int u\partial_x^2u(1-\chi(x/M))\,dx = -\varepsilon\int u_x^2(1-\chi(x/M))\,dx - \frac{\varepsilon}{2M^2}\int u^2\chi''(x/M)\,dx\\
&\varepsilon\int u_x\partial_x^3u(1-\chi(x/M))\,dx = -\varepsilon\int u_{xx}^2(1-\chi(x/M))\,dx - \frac{\varepsilon}{2M^2}\int u_x^2\chi''(x/M)\,dx\ .
\end{aligned}
$$

Putting all the terms together yields

$$
\begin{aligned}
J'(t) = &-\frac{1}{M}\int u_x^2u\chi'(x/M)\,dx - \frac{1}{M}\int u(1-\partial_x^2)^{-1}(u_x^2)\chi'(x/M)\,dx\\
&-\frac{2}{M}\int u(1-\partial_x^2)^{-1}(u^2)\chi'(x/M)\,dx\\
&-\varepsilon\int(u_x^2+u_{xx}^2)(1-\chi(x/M))\,dx - \frac{\varepsilon}{2M^2}\int(u^2+u_x^2)\chi''(x/M)\,dx\ .
\end{aligned}
$$

Estimating each term by Hölder's inequality and by means of Lemma 1 yields

$$\left|\int u_x^2 u \chi'(x/M)\,dx\right| \le C\|u_x\|_{L^2}^2\|u\|_{L^\infty} \le C\|u\|_{H^1}^3$$

$$\left|\int u\big(1-\partial_x^2\big)^{-1}\big(u_x^2\big)\,dx\right| \le C\|u\|_{L^2}\|u_x\|_{L^2}^2 \le C\|u\|_{H^1}^3$$

$$-\varepsilon\int \big(u_x^2+u_{xx}^2\big)(1-\chi(x/M))\,dx \le 0$$

$$\left|\int \big(u^2+u_x^2\big)\chi''(x/M)\,dx\right| \le C\|u\|_{H^1}^2\,,$$

which implies the lemma for the case of (3.1). Let us now focus on the solutions of the frequency localized version (4.2). Compute $J'(t)$ as in the case of (3.1). We get

$$\begin{aligned} J'(t) = & -2\int uP_{<N}F(P_{<N}u,\,P_{<N}u_x)(1-\chi(x/M))\,dx \\ & -2\int u_x\partial_x P_{<N}F(P_{<N}u,\,P_{<N}u_x)(1-\chi(x/M))\,dx\,, \end{aligned}$$

where F is the nonlinearity of the Camassa-Holm equation. Conjugating $P_{<N}$ yields

$$\begin{aligned} J'(t) = & -2\int F(P_{<N}u,\,P_{<N}u_x)P_{<N}(u(1-\chi(x/M)))\,dx \\ & -2\int \partial_x P_{<N}F(P_{<N}u,\,P_{<N}u_x)P_{<N}(u_x(1-\chi(x/M)))\,dx \\ = & -2\int (P_{<N}u)F(P_{<N}u,\,P_{<N}u_x)(1-\chi(x/M))\,dx \\ & -2\int (P_{<N}u_x)\partial_x P_{<N}F(P_{<N}u,\,P_{<N}u_x)(1-\chi(x/M))\,dx \\ & -2\int F(P_{<N}u,\,P_{<N}u_x)[P_{<N},\,(1-\chi(x/M))]u\,dx \\ & -2\int \partial_x P_{<N}F(P_{<N}u,\,P_{<N}u_x)[P_{<N},\,(1-\chi(x/M))]u_x\,dx\,. \end{aligned}$$

The first two terms in the expression above are controlled in exactly the same way as in the case of (3.1). Indeed, one only need to replace u by $P_{<N}u$ in that argument to achieve the same results. The estimates in the end are in terms of $\|P_{<N}u\|_{H^1}$, which is clearly bounded by $\|u\|_{H^1}$. The remaining two terms can be considered as error terms. Note first that $P_{<N}$ is well-defined Calderón-Zygmund operator with Calderón-Zygmund bounds independent of N. Taking into account the form of F, we get

$$\begin{aligned} & \left|\int F(P_{<N}u,\,P_{<N}u_x)[P_{<N},\,(1-\chi(x/M))]u\,dx\right| \\ & = \Big|\int \Big((P_{<N}u)^2/2+\big(1-\partial_x^2\big)^{-1}(P_{<N}u_x)^2/2+\big(1-\partial_x^2\big)^{-1}(P_{<N}u)^2\Big) \\ & \qquad \times \partial_x[P_{<N},\,(1-\chi(x/M))]u\,dx\Big| \\ & \le \Big\|(P_{<N}u)^2/2+\big(1-\partial_x^2\big)^{-1}(P_{<N}u_x)^2/2+\big(1-\partial_x^2\big)^{-1}(P_{<N}u)^2\Big\|_{L^2} \\ & \qquad \times \|\partial_x[P_{<N},\,(1-\chi(x/M))]u\|_{L^2} \le CM^{-1}\big\|\chi'\big\|_{L^\infty}\|u\|_{H^1}^2\|u\|_{L^2}\,, \end{aligned}$$

where in the last inequality, we have used Lemma 1, the Sobolev embedding and Lemma 2. Similarly, we estimate

$$\begin{aligned}
&\left|\int \partial_x F(P_{<N}u, P_{<N}u_x)[P_{<N}, (1-\chi(x/M))]u_x\, dx\right| \\
&= \Big|\int \left(\partial_x (P_{<N}u)^2/2 + \partial_x\left(1-\partial_x^2\right)^{-1}(P_{<N}u_x)^2/2 + \partial_x\left(1-\partial_x^2\right)^{-1}(P_{<N}u)^2\right) \\
&\quad \times \partial_x[P_{<N}, (1-\chi(x/M))]u_x\, dx\Big| \\
&\le \left\|\partial_x (P_{<N}u)^2/2 + \partial_x\left(1-\partial_x^2\right)^{-1}(P_{<N}u_x)^2/2 + \partial_x\left(1-\partial_x^2\right)^{-1}(P_{<N}u)^2\right\|_{L^2} \\
&\quad \times \|\partial_x[P_{<N}, (1-\chi(x/M))]u_x\|_{L^2} \le CM^{-1}\left\|\chi'\right\|_{L^\infty}\|u\|_{H^1}^2\|u_x\|_{L^2}\ ,
\end{aligned}$$

by the same arguments. Lastly, by the energy dissipation for the equation, we have that $\|u\|_{H^1} \le \|u_0\|_{H^1}$ and Lemma 5 follows. □

7.2 Uniform Vanishing of the Fourier Transform Implies Equicontinuity

In this section, we show that for a sequence of uniformly bounded functions, uniform vanishing of the Fourier transforms (in $L^{2,2s}$) implies (and in fact is equivalent) to equicontinuity.

Lemma 6. *Let $s \ge 0$ and $\{u_n\}$ be sequence of functions in H^s with*

- $\sup_n \|u_n\|_{H^s} \le C < \infty$
- $\sup_n \int_{|\xi|>M} |\widehat{u_n}(\xi)|^2 <\xi>^{2s}\ d\xi \to 0$ *as* $M \to \infty$.

Then

$$\sup_n \|u_n(\cdot + h) - u_n(\cdot)\|_{H^s} \to 0 \qquad \text{as} \qquad h \to 0\ .$$

Proof. Take $\varepsilon > 0$ and select $M = M(\varepsilon)$ so that

$$\sup_n \int_{|\xi|>M} \left|\widehat{u_n}(\xi)\right|^2 |\xi|^{2s}\, d\xi \le \varepsilon^2\ .$$

Choose $\delta = \varepsilon/(MC)$. For every $|h| \le \delta$, we have by the Plancherel's theorem

$$\begin{aligned}
\sup_n \|u_n(\cdot + h) - u_n(\cdot)\|_{H^s}^2 &\le \int \left|e^{2\pi i h\xi} - 1\right|^2 \left|\widehat{u_n}(\xi)\right|^2 <\xi>^{2s}\ d\xi \\
&\lesssim \int_{|\xi|>M} \left|\widehat{u_n}(\xi)\right|^2 <\xi>^{2s}\ d\xi + h^2\int_{|\xi|<M} |\xi|^2\left|\widehat{u_n}(\xi)\right|^2 <\xi>^{2s}\ d\xi \\
&\lesssim \varepsilon^2 + h^2M^2 \sup_n \|u_n\|_{H^s} \le 2\varepsilon^2\ .
\end{aligned}$$

□

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Department of Mathematics, University of KS, 1460 Jayhawk Blvd, Lawrence, KS 66045–7523
e-mail: stanis@math.ku.edu

Department of Mathematics, University of Kansas, 405 Snow Hall 1460 Jayhawk Blvd,
Lawrence, KS 66045-7567
e-mail: stefanov@math.ku.edu