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# ON THE SPECTRAL STABILITY OF GROUND STATES OF SEMI-LINEAR SCHRÖDINGER AND KLEIN-GORDON EQUATIONS WITH FRACTIONAL DISPERSION

*Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday*

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Abstract. We consider standing wave solutions of various dispersive models with non-standard form of the dispersion terms. Using index count calculations, together with the information from a variational construction, we develop sharp conditions for spectral stability of these waves.

## 1. Introduction and statement of the main results.

For $s \in (0,1]$ and $d \geq 1$, we consider the focusing fractional Schrödinger equation

$$iu_t - (-\Delta)^s u + |u|^\alpha u = 0, (t,x) \in \mathbf{R}_+ \times \mathbf{R}^d \tag{1}$$

In addition, we shall be interested in the fractional Klein-Gordon equation

$$u_{tt} + (-\Delta)^s u + u - |u|^\alpha u = 0, (t,x) \in \mathbf{R} \times \mathbf{R}^d \tag{2}$$

These nonlocal equations arise in a variety of models in mathematical physics, see many examples in [1] and the references therein. Also, a similar model

$$iu_t + (-\Delta)^s u + |u|^\alpha u = 0, \tag{3}$$

has been introduced by Laskin in quantum physics [17], and it is a fundamental equation of fractional quantum mechanics, a generalization of the standard quantum mechanics extending the Feynman path integral to Levy processes[17]. Further, in [10], Hong and Sire have discussed the local well-posedness and ill-posedness in Sobolev spaces, and in [9], Guo and Huo focused on the global well-posedness for the Cauchy problem of the 1-D fractional nonlinear Schrödinger equation with data in $L^2(\mathbf{R})$. Regarding well-posedness in the natural energy space, one has local and hence global well-posedness for Cauchy data in $H^s(\mathbf{R}^d)$, provided $\alpha < \frac{4s}{d}$, due to the conservation law. Generally, some solutions will blow up for $\alpha > \frac{4s}{d}$, [5].

Additionally, we will be interested in two higher order dispersion models, which are outside of the scope of (1) and (2). Namely, we consider the fourth order cubic

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Schrödinger equation, in one spatial dimension

$$iu_t + u_{xx} - u_{xxxx} + |u|^2 u = 0 \tag{4}$$

and the fourth order cubic Klein-Gordon equation

$$u_{tt} + u_{xxxx} - u_{xx} + u - |u|^2 u = 0, \tag{5}$$

The fourth order Schrödinger equation was introduced in [14] and [15], and it has an important role in modeling the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Moreover, the equation was also used in nonlinear fiber optics and the theory of optical solitons in gyrotropic media. In this paper, we are interested in the existence and linear stability of standing wave solutions for these equations.

1.1. **Standing wave solutions for fractional models.** The existence of special solutions is an important feature of the fractional models. More precisely, we seek solutions of the fractional NLS equation (i.e. (1)) in the form $u_\omega(t,x) = e^{i\omega t}Q_\omega(x), \omega > 0$, with $Q_\omega > 0$. We obtain the following profile equation

$$\omega Q_\omega + (-\Delta)^s Q_\omega - Q_\omega^{\alpha+1} = 0, x \in \mathbf{R}^d \tag{6}$$

For the fractional Klein-Gordon equation, we have the profile equation

$$(1-\omega^2)R_\omega + (-\Delta)^s R_\omega - R_\omega^{\alpha+1} = 0, x \in \mathbf{R}^d. \tag{7}$$

where we require that $|\omega| < 1$, $R_\omega > 0$. Clearly (6) and (7) are closely related to each other. Indeed, setting for each $\omega \in (-1,1)$, $\gamma := 1-\omega^2 > 0$, whence $R_\omega = Q_\gamma$. Thus, we proceed to describe the properties of $Q_\omega$, keeping in mind this relationship.

Note that the equation (6) enjoys a nice scaling property, which allows one to explicitly describe the solutions $Q_\omega$ of (6) in terms of a single function. To this end, consider (6) with $\omega = 1$,

$$(-\Delta)^s Q + Q - Q^{\alpha+1} = 0, x \in \mathbf{R}^d, \tag{8}$$

where we henceforth adopt for brevity the notation $Q = Q_1$. If one establishes that (8) has a unique (modulo symmetries) solution $Q$, then all solutions of (6) (modulo symmetries) are given by the formula $Q_\omega = w^{\frac{1}{\alpha}} Q(w^{\frac{1}{2s}} x)$.

The equation (8) has been well-studied, at least in the classical case $s = 1$, in the last thirty years. First, it is well-known that for $s = 1, d = 1, \alpha > 0$, such solutions are explicitly given in terms of powers of the sech functions. Clearly, one cannot hope for such solutions to be explicit outside of the cases mentioned above. In the case $s = 1$, $d \geq 1$, $\alpha > 0$, it has been shown in the classical paper [16] that such $Q : Q > 0$ is unique, modulo the translational symmetries. In the fractional case, i.e. $s \in (0,1)$, this difficult problem was resolved recently. It has been shown (in [7] for the case $d = 1$ and subsequently in [8] for the case $d \geq 2$) that (8) possesses a unique positive radial solution[1], provided

$$0 < \alpha < \alpha_*(s,d) = \begin{cases} \dfrac{4s}{d-2s} & s < \dfrac{d}{2} \\ \infty & s > \dfrac{d}{2}. \end{cases}$$

On the other hand, Pokhozaev type arguments for the elliptic equation (8) show that smooth and localized solutions $Q$ do not exist, when $\alpha > \alpha_*(s,d)$.

[1] which we refer to, with a slight abuse of notation, by $Q(|x|)$

In addition to the uniqueness, a number of additional properties of $Q$ were established, which will be important for us as well and we discuss them below. The main tool in establishing all these important results has been the heavy use of the fact that a variant of (8) is in fact the Euler-Lagrange equation of a particular constrained minimization problem and $Q$ is its minimizer.

1.2. **Standing waves for fourth order models.** It is clear that the fourth order case (which roughly corresponds to the case $s = 2, d = 1$ of our fractional family of equations) does not fit in the Frank-Lenzmann theory. Indeed, important ingredients of their proofs break down, such as maximum principle and positivity of the heat kernels of the corresponding semigroups, to mention a few. Nevertheless, it is an interesting question whether there exist any reasonable solutions of the profile equations and if so, what are their stability properties. More precisely, we again consider solutions in the form $u = e^{i\alpha t}\phi$ of (4), which yields the profile equation

$$\phi'''' - \phi'' + \alpha\phi - \phi^3 = 0. \tag{9}$$

The ansatz $\phi(x) = asech^2(bx)$ produces, for $\alpha = \frac{4}{25}$, the solution

$$\phi(x) = \sqrt{\frac{3}{10}} sech^2\left(\frac{x}{\sqrt{20}}\right). \tag{10}$$

Here, the solution displayed in (10) serves as a standing wave to the fourth order Schrödinger equation (4). A simple modification provides a solution to the fourth order Klein-Gordon equation as well. Indeed, a direct verification shows that

$$u(x,t) = e^{i\frac{\sqrt{21}}{5}t}\phi(x) \tag{11}$$

is a solution to (5). One of the main difficulties associated with the stability analysis of (10) ( (11) respectively) is the fact that no explicit solution is available for values of $\alpha \neq \frac{4}{25}$. In other words, since we lack a one parameter family of solutions, the spectral computations become quite delicate. In particular, the standard approach to computing certain quantities related to stability depends on taking a derivative (in the explicit solution) in terms of $\alpha$. This is the usual presentation of the Vakhitov-Kolokolov criteria, which in this case necessarily fails, due to the fact that such an explicit formula in terms of $\alpha$ is simply unavailable. We overcome these issues by resorting to the positivity theory as developed in [2, 3, 4].

In the next sections, we consider the linearized problems associated with the stability of these solitary waves.

1.3. **The eigenvalue problem for the fractional NLS model.** We first linearize around the standing wave $Q = Q_1$ in (1). Using the ansatz

$$u = e^{it}\{Q + (\varphi + i\psi)\}, \tag{12}$$

and taking real and imaginary parts leads us to

$$\begin{aligned} \varphi_t &= (-\Delta)^s\psi + \psi - Q^\alpha\psi \\ -\psi_t &= (-\Delta)^s\varphi + \varphi - (\alpha+1)Q^\alpha\varphi. \end{aligned}$$

Introduce the skew symmetric portion is $\mathcal{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the self-adjoint portion of the linearized operator $\mathcal{L} := \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$ with

$$\begin{aligned} L_1 &= (-\Delta)^s + 1 - (\alpha+1)Q^\alpha, \\ L_2 &= (-\Delta)^s + 1 - Q^\alpha. \end{aligned}$$

both acting on the domain $H^{2s}(\mathbf{R}^d)$. It is now clear that the eigenvalue problem is in the form

$$\partial_t \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \mathcal{J}\mathcal{L} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \tag{13}$$

Standard scaling argument shows that stability for $Q_1$ is equivalent to the stability for $Q_\omega, \omega > 0$, whence we henceforth concentrate on this particular case.

1.4. **The eigenvalue problem for the fractional Klein-Gordon model.** For the fractional KG model, (2), we linearize at the solution $e^{iwt}(1-w^2)^{\frac{1}{\alpha}} Q((1-w^2)^{\frac{1}{2s}} x)$. More precisely, we take the ansatz

$$u = (1-w^2)^{\frac{1}{\alpha}} e^{iwt} \{Q((1-w^2)^{\frac{1}{2s}} x) + v((1-w^2)^{\frac{1}{2s}} x, t)\},$$

Ignoring all second and higher order terms leads us to the eigenvalue problem

$$\begin{aligned} iwv_t &+ v_{tt} - w^2(Q+v) + (1-w^2)(-\Delta)^s Q + \\ &+ ((1-w^2)(-\Delta)^s v + Q + v - (1-w^2)(Q^{\alpha+1} + Q^\alpha v + \alpha Q^\alpha \Re(v)) = 0, \end{aligned}$$

Letting $v = \begin{pmatrix} \Re v \\ \Im v \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ allows us to rewrite the eigenvalue problem in the following matrix form

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_{tt} + \begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}_t + (1-w^2)\mathcal{L} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0 \tag{14}$$

where $\mathcal{L}$ is already defined in (13). Equivalently, writing $\varphi \to e^{\lambda t}\varphi, \psi \to e^{\lambda t}\psi$, one can write the last second order model as a first order system in the form

$$\partial_t \begin{pmatrix} \varphi \\ \psi \\ \varphi_t \\ \psi_t \end{pmatrix} = \mathcal{J}\mathcal{L} \begin{pmatrix} \varphi \\ \psi \\ \varphi_t \\ \psi_t \end{pmatrix}, \tag{15}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & w \\ 0 & -1 & -w & 0 \end{pmatrix}, \mathcal{L} = \begin{pmatrix} (1-w^2)L_1 & 0 & 0 & 0 \\ 0 & (1-w^2)L_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

are a skew-symmetric and a self-adjoint operators respectively.

1.5. **The eigenvalue problem of the fourth order models.** We now derive the relevant eigenvalue problem for the fourth order Schrödinger model (4).

In order to consider the stability of the wave $e^{i\alpha t}\phi$, with $\alpha = \frac{4}{25}$ and $\phi$ given by (10). We take

$$u = e^{i\alpha t}[\phi + v + iw], \tag{17}$$

for real-valued functions $v, w$ and plug it into (4). Ignoring the contributions of terms in the form $O(v^2), O(w^2)$ and some algebra leads us to the the eigenvalue problem

$$\partial_t \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x^4 - \partial_x^2 + \alpha - 3\phi^2 & 0 \\ 0 & \partial_x^4 - \partial_x^2 + \alpha - \phi^2 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \quad (18)$$

As usual, we denote $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathcal{L} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$ with

$$\begin{cases} L_1 = & \partial_x^4 - \partial_x^2 + \alpha - 3\phi^2, \\ L_2 = & \partial_x^4 - \partial_x^2 + \alpha - \phi^2. \end{cases} \quad (19)$$

Finally, we discuss the linearization (and subsequently the eigenvalue problem) associated with the solution (11) to the fourth order cubic equation (5). To introduce proper notations, let $\beta = \frac{\sqrt{21}}{5}$, so that the wave is exactly $e^{i\beta t}\phi(x) = e^{i\beta t}\sqrt{\frac{3}{10}} sech^2(\sqrt{\frac{1}{20}}x)$. Setting

$$u = e^{i\beta t}(\phi + \varphi + i\psi),$$

plugging this ansatz into (5), ignoring the contributions of the type $O(\varphi^2), O(\psi^2)$ and taking real and imaginary parts, we obtain

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}_{tt} + \begin{pmatrix} 0 & -2\beta \\ 2\beta & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}_t + \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0, \quad (20)$$

where $L_1, L_2$ are exactly the operators introduced in (19). We can also write a further equivalent formula

$$\partial_t \begin{pmatrix} \varphi \\ \psi \\ \varphi_t \\ \psi_t \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -L_1 & 0 & 0 & 2\beta \\ 0 & -L_2 & -2\beta & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \\ \varphi_t \\ \psi_t \end{pmatrix} =: \mathcal{H} \begin{pmatrix} \varphi \\ \psi \\ \varphi_t \\ \psi_t \end{pmatrix} \quad (21)$$

We note that in addition

$$\mathcal{H} = \mathcal{J}\mathcal{L} = \begin{pmatrix} 0 & I_2 \\ -I_2 & B \end{pmatrix} \begin{pmatrix} \widetilde{\mathcal{L}} & 0 \\ 0 & I_2 \end{pmatrix} \quad (22)$$

$$\widetilde{\mathcal{L}} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, B = \begin{pmatrix} 0 & -2\beta \\ 2\beta & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (23)$$

1.6. **Main results.** We are now ready to state our results, first for the fractional NLS.

**Theorem 1.** *The standing waves $e^{i\omega t}Q_\omega$ of the fractional NLS (1) are linearly and orbitally stable for $\alpha < \frac{4s}{d}$. Moreover, they are linearly unstable for $\alpha > \frac{4s}{d}$.*

*For the fractional Klein-Gordon model, the soliton $e^{i\omega t}(1-\omega^2)^{\frac{1}{\alpha}}Q((1-\omega^2)^{\frac{1}{2s}}x)$ is spectrally stable if and only if*

$$\alpha < \frac{4s}{d}, \quad \sqrt{\frac{4s\alpha}{4s\alpha + 4s - \alpha d}} < |\omega| < 1$$

Our next result concerns the stability of the waves for the fourth order Schrödinger and Klein-Gordon equations.

**Theorem 2.** *The wave $e^{i\alpha t}\phi$ (with $\alpha = \frac{4}{25}$ and $\phi$ given by (10)) is spectrally stable solution of (4). The wave $e^{i\beta t}\phi$, with $\beta = \frac{\sqrt{21}}{5}$ is spectrally unstable as a solution to the fourth order Klein-Gordon model (5).*

The plan of the paper is as follows - in Section 2, we introduce first the instability index counting theory, which will be the main theoretical tool for us. Then, we describe the relevant spectral theory of the self-adjoint portion of the linearized operators. In Section 3, we apply the instability index count to the standing waves of the fractional NLS and Klein-Gordon models and derive sharp conditions for their spectral stability. In Section 4, we use the theory developed by J. Albert, [2] (presented in some details in the Appendix) in order to obtain the necessary spectral information for the self-adjoint pieces of the linearized operators. Using the index count, the spectral stability of the solutions is reduced again to a sign condition for a Vakhitov-Kolokolov type quantities. These are computed explicitly as infinite series, again by following some ideas by Albert, [2], presented in Section B.2.

## 2. Preliminaries.

We start by outlining the instability index count theory, as developed in [11, 12]. We will constrain ourselves to a simple, yet representative corollary, which suffices for our purposes. We consider the eigenvalue problem in the form

$$\mathcal{J}\mathcal{L}f = \lambda f, \tag{24}$$

where $\mathcal{J}$ is assumed to be bounded, invertible and skew-symmetric ($\mathcal{J}^* = -\mathcal{J}$), while $(\mathcal{L}, D(\mathcal{L}))$ is self-adjoint($\mathcal{L}^* = \mathcal{L}$) and not necessarily bounded, with finite dimensional kernel $Ker[\mathcal{L}]$. In addition, we assume that $\mathcal{L}$ has a finite number of negative eigenvalues, $n(\mathcal{L})$ and $\mathcal{J}^{-1} : Ker[\mathcal{L}] \to Ker[\mathcal{L}]^{\perp}$. Here, the orthogonality is understood with respect to the dot product of the underlying Hilbert space $H$ : $D(\mathcal{L}) \subset H$.

Let $k_r$ denote the number of positive eigenvalues of (24), $k_c$ be the number of quadruplets of eigenvalues with non-zero real and imaginary parts, and $k_i^-$, the number of pairs of purely imaginary eigenvalues with negative Krein-signature[2]. Introduce the matrix $D$ as follows. Let $Ker[\mathcal{L}] = \{\phi_1, \dots, \phi_n\}$, then

$$D_{ij} := \langle \mathcal{L}^{-1}[\mathcal{J}^{-1}\phi_i], \mathcal{J}^{-1}\phi_j \rangle. \tag{25}$$

Note that the last formula makes sense, since $\mathcal{J}^{-1}\phi_i \in Ker[\mathcal{L}]^{\perp}$. Thus $\mathcal{L}^{-1}[\mathcal{J}^{-1}\phi_i]$ is well-defined. The index counting theorem, see Theorem 1, [12] states that if $det(D) \neq 0$, then

$$k_r + 2k_c + 2k_i^- = n(\mathcal{L}) - n(D). \tag{26}$$

### 2.1. Spectral information regarding the operators $L_1, L_2$.

By the representations of the Hamiltonian in both (13) and (16), it is clear that the spectral properties of the operators $L_1, L_2$ will play substantial role in our analysis.

**Proposition 1.** *The operator $L_1$ defined in (13) has a unique negative eigenvalue, which is simple. The eigenvalue zero is of multiplicity $d$, with $Ker[L_1] = span\{\partial_1 Q, \dots, \partial_d Q\}$. The operator $L_2$ satisfies $L_2 \geq 0$, with an eigenvalue at zero, which corresponds to the eigenfunction $Q$. As such the eigenvalue at zero is simple. Moreover, the essential spectrum for both operators is $[1, \infty)$.*

*Proof.* For $L_1$, we refer to the paper [8], where it was shown that $n(L_1) = 1$, while $Ker[L_1] = \{\partial_1 Q, \dots, \partial_d Q\}$.

Next, we clearly have $L_2[Q] = 0$, by construction of $Q$. We now show that $L_2$ has no negative eigenvalues. Assume for a contradiction that $L_2$ has a negative

[2] The precise definition of those is provided in many references, for example in [11]. This will be irrelevant for us, since in our applications $k_i^- = 0$

eigenvalue, say we pick the smallest such eigenvalue $-\sigma^2$. Then, there is an $F \neq 0$, so that $L_2[F] = -\sigma^2 F, \|F\| = 1$. According to the Rayleigh characterization of e-values,

$$-\sigma^2 = \inf_{\|G\|=1} \langle L_2 G, G\rangle$$

and so $F$ is a solution of this problem. Rewrite this constrained minimization problem in the form

$$\begin{cases} \langle L_2 G, G\rangle = \|(-\Delta)^{s/2} G\|_{L^2}^2 + \|G\|_{L^2}^2 - \int_{\mathbf{R}^d} Q^\alpha(x) G^2(x) dx \to \min \\ \int_{\mathbf{R}^d} G^2(x) dx = 1 \end{cases} \tag{27}$$

We now need to refer to recent results on the multi-dimensional Polya-Szegö inequality, which imply that the functional $\langle L_2 G, G\rangle$ is minimized by its decreasing rearrangement. More precisely, for $s \in (0,1)$, there is the generalized Polya-Szegö inequality[3]

$$1\|(-\Delta)^{s/2} G\|_{L^2} \geq \|(-\Delta)^{s/2} G^*\|_{L^2},$$

where $G^*$ is the decreasing rearrangement of the function $G$. Moreover, since $Q^\alpha$ is radially decreasing, there is the simple inequality

$$\int_{\mathbf{R}^d} Q^\alpha(x) G^2(x) dx \leq \int_{\mathbf{R}^d} Q^\alpha(x) [G^*]^2(x) dx \tag{28}$$

where the equality in (28) is achieved only if $G = G^*$. This is a simple consequence of $\int fg \leq \int f^* g^*$, see Theorem 3.4, [18]. In addition, an elementary property of the decreasing rearrangement says that $\|G\|_{L^p} = \|G^*\|_{L^p}$ for $p \in (0,\infty)$ and in particular for $p = 2$. It follows that $\langle L_2 G, G\rangle \geq \langle L_2 G^*, G^*\rangle$, with equality possible only if $G = G^*$, while clearly $\|G\|_{L^2} = \|G^*\|_{L^2}$. Thus, the eigenfunction $F$, corresponding to the lowest eigenvalue $-\sigma^2$ must necessarily be such that $F = F^*$ ( since it isa solution to the constrained minimization problem (27)). In particular $F \geq 0$. But then $\langle F, Q\rangle = 0$, since any two e-functions corresponding to two different eigenvalues of $L_2$ must be orthogonal. This however leads to a contradiction, since $F \geq 0$, $Q > 0$. Thus, 0 is the lowest eigenvalue for $L_2$, whence $L_2 \geq 0$. □

3. **On the stability of the standing waves for the fractional NLS and Klein-Gordon equations.** We consider the cases of NLS and Klein-Gordon separately, although there is quite a few calculations that will appear in both.

3.1. **Stability of fNLS waves.** For the stability of the eigenvalue problem (13), we take the standard transformation $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \to e^{\lambda t} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$, which puts us in the form (24). In addition, due to the results of Proposition 1, the self-adjoint operator $\mathcal{L}$ satisfies $n(\mathcal{L}) = 1$ and

$$Ker[\mathcal{L}] = \left\{ \begin{pmatrix} 0 \\ Q \end{pmatrix}, \begin{pmatrix} \partial_1 Q \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} \partial_d Q \\ 0 \end{pmatrix} \right\} =: \{Q_0, Q_1, \ldots, Q_d\}. \tag{29}$$

In addition, it is clear that $\mathcal{J}^{-1} = -\mathcal{J} : Ker[\mathcal{L}] \to Ker[\mathcal{L}]^\perp$, whence the matrix $D \in \mathcal{M}_{(d+1)\times(d+1)}$ may be defined as in (25). Obviously, for $j \geq 1$, $D_{0j} = 0$. Next, note that for $i \geq 1, j \geq 1, i \neq j$, we have

$$D_{ij} = \langle L_2^{-1} \partial_i Q, \partial_j Q\rangle = 0, \tag{30}$$

[3] for which one can consult the recent work [7] or the direct and easy proof, which we provide in the Appendix, Proposition 3

since $\partial_i Q$ is odd in the $i^{th}$ variable (and then so[4] is $L_2^{-1}[\partial_i Q]$), while $\partial_j Q$ is odd in the $j^{th}$ variable. On the other hand, for $i = 1, \dots d$,

$$D_{ii} = \langle L_2^{-1} \partial_i Q, \partial_i Q \rangle > 0,$$

due to the positivity of $L_2^{-1}$ on $Ker[L_2]^\perp$ and the fact that $\partial_i Q \perp Ker[L_2] = span[Q]$. Clearly now, $n(D) = n(\langle \mathcal{L}^{-1} \mathcal{J}^{-1} Q_0, \mathcal{J}^{-1} Q_0 \rangle = n(\langle L_1^{-1}[Q], Q \rangle)$.

In order to compute this quantity, we use the standard scaling properties of the profile equation (8). Namely, $Q_\mu := \mu^{\frac{1}{\alpha}} Q(\mu^{\frac{1}{2s}} x)$ solves

$$\mu Q_\mu + (-\Delta)^s Q_\mu - Q_\mu^{\alpha+1} = 0.$$

Taking derivative in $\mu$ yields the relation $L_1[\frac{\partial Q_\mu}{\partial \mu}] = -Q_\mu$, whence since $Q_\mu \perp Ker[L_1]$, we derive $L_1^{-1}[Q_\mu] = -\frac{\partial Q_\mu}{\partial \mu}$, whence

$$\langle L_1^{-1}[Q], Q \rangle = -\frac{1}{2} \partial_\mu \|Q_\mu\|^2|_{\mu=1} = -\frac{1}{2} \left( \frac{2}{\alpha} - \frac{d}{2s} \right) \|Q\|^2. \tag{31}$$

In view of (26), the fact that $n(\mathcal{L}) = 1$, the spectral stability of fNLS waves is equivalent to $\langle L_1^{-1}[Q], Q \rangle < 0$, or $\frac{2}{\alpha} - \frac{d}{2s} > 0$. This is easily seen to be equivalent to $\alpha < \frac{4s}{d}$ as stated. Due to the structure of $Ker[\mathcal{L}]$, namely (29), all the elements of the $Ker[\mathcal{L}]$ are accounted for by invariances of the model, so by the results of [13] (Theorem 5.2.11) and the well-posedness of the Cauchy problem in the energy space $H^s(\mathbb{R}^d)$ established in [6], the waves are orbitally stable as well.

3.2. **Stability of the fKG waves.** The relevant eigenvalue problem for the stability of the fractional Klein-Gordon waves is $\mathcal{J} \mathcal{L} \vec{X} = \lambda \vec{X}$, where $\mathcal{J}, \mathcal{L}$ are given by (16). By the form of $\mathcal{L}$, we have that $n(\mathcal{L}) = n(L_1) = 1$, owing to Proposition 1. The description of $Ker[\mathcal{L}]$ is again explicit, thanks again to Proposition 1. More precisely, we have

$$Ker[\mathcal{L}] = \{Q_0, Q_1, \dots, Q_d\}, Q_0 = \begin{pmatrix} 0 \\ Q \\ 0 \\ 0 \end{pmatrix}, Q_j = \begin{pmatrix} \partial_j Q \\ 0 \\ 0 \\ 0 \end{pmatrix}, j = 1, \dots, d.$$

Since $\mathcal{J}^{-1} = \begin{pmatrix} 0 & \omega & -1 & 0 \\ -\omega & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, we have by (25), that for $i \geq 1, j \geq 1, i \neq j$,

$$D_{ij} = \frac{\omega^2}{1-\omega^2} \langle L_2^{-1}[\partial_i Q], \partial_j Q \rangle = 0,$$

by (30). Similarly, $D_{i0} = D_{0i} = 0$ by our previous arguments for the fNLS case. Thus, the matrix $D$ has only diagonal potentially non-zero entries. In fact, the entries $D_{ii}, i = 1, \dots, n$ are positive due to the positivity of $L_2^{-1}$ on $Ker[L_2]^\perp$.

[4]Note that the space of functions which are odd in the $j^{th}, j = 1, \dots, d$ variable, is an invariant subspace for $L_2^{-1}$

Indeed,

$$
\begin{aligned}
D_{ii} &= \langle \mathcal{L}^{-1}\mathcal{J}^{-1}Q_i, \mathcal{J}^{-1}Q_i\rangle = \\
&= \langle \begin{pmatrix} \frac{L_1^{-1}}{1-\omega^2} & 0 & 0 & 0 \\ 0 & \frac{L_2^{-1}}{1-\omega^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -\omega\partial_i Q \\ \partial_i Q \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\omega\partial_i Q \\ \partial_i Q \\ 0 \end{pmatrix} \rangle = \\
&= \frac{\omega^2}{1-\omega^2}\langle L_2^{-1}[\partial_i Q], \partial_i Q\rangle + \|\partial_i Q\|^2 > 0.
\end{aligned}
$$

Thus, as before, matters have been reduced to $D_{00}$, more precisely, $n(D) = n(D_{00})$. The stability condition, according to (26) is exactly $D_{00} < 0$. We have, according to (31)

$$
\begin{aligned}
D_{00} &= \langle \mathcal{L}^{-1}\mathcal{J}^{-1}Q_0, \mathcal{J}^{-1}Q_0\rangle = \frac{\omega^2}{1-\omega^2}\langle L_1^{-1}[Q], Q\rangle + \|Q\|^2 = \\
&= \left[\frac{\omega^2}{1-\omega^2}\left(\frac{d}{4s} - \frac{1}{\alpha}\right) + 1\right]\|Q\|^2.
\end{aligned}
$$

It is clear that the stability condition is satisfied only if $\alpha < \frac{4s}{d}$ and then,

$$
\frac{\omega^2}{1-\omega^2} > \frac{4s\alpha}{4s-\alpha d}.
$$

Resolving this last inequality yields the condition

$$
\omega^2 > \frac{4s\alpha}{4s\alpha + 4s - \alpha d}.
$$

Since we have initially required $|\omega| < 1$ for the existence of the waves, we can finally formulate the necessary and sufficient condition for stability as follows

$$
\frac{4s\alpha}{4s\alpha + 4s - \alpha d} < \omega^2 < 1.
$$

Note that this last inequality implicitly requires $\alpha < \frac{4s}{d}$, since otherwise the double inequality will not have any solutions in $\omega$.

4. **On the stability of the standing waves for the fourth order models.** We start this section with a discussion about the spectral properties of the self-adjoint operators $L_1, L_2$, defined in (19). We have the following result.

**Proposition 2.** *The operator $L_1$ with domain $H^4(\mathbb{R})\times H^4(\mathbb{R})$ has a unique negative eigenvalue, which is simple. The eigenvalue zero is of dimension exactly $d = 1$, with associated eigenfunctions $\partial_j\phi, j = 1, \dots d$. $L_2$ has no negative eigenvalues, it has eigenvalue at zero, which is simple. Moreover the essential spectrum is the interval $[\alpha, \infty)$.*

**Note:** The results about $L_1$ stated here have been established in [2], in relation to a model of water wave equations with non-standard dispersions. The result about $L_2$ is a minor modification of these arguments, we present it below in Appendix B. Having the results of Proposition 2 allows us to go through the index counting (26).

4.1. **Stability of the wave of the fourth order Schrödinger equation (4).** Matters are reduced to the number of negative eigenvalues of $D$, introduced in (25). As we have previously observed on the related fractional NLS model,

$$D = \begin{pmatrix} \langle L_2^{-1}\phi', \phi' \rangle & 0 \\ 0 & \langle L_1^{-1}\phi, \phi \rangle \end{pmatrix},$$

which in view of the positivity of $L_2^{-1}$ on $Ker[L_2]^\perp$ reduces to the consideration of the quantity $\langle L_1^{-1}\phi, \phi\rangle$. The stability is then characterized by the condition $\langle L_1^{-1}\phi, \phi\rangle < 0$. Recalling that $L_1 = \partial_x^4 - \partial_x^2 + \alpha - 3\phi^2$, with $\phi$ given by (10), we apply the Albert's theory for the quantity $\langle L_1^{-1}\phi, \phi\rangle$, see Section B.2 and (37) below. More specifically, in the notations there, we take $n = 2, r = 2, p = 2$, which yields the formula

$$\lambda_{2j} = \frac{\Gamma(2j+2)}{\Gamma(3)} \cdot \frac{\Gamma(7)}{\Gamma(2j+6)} = \frac{6!}{2!} \cdot \frac{(2j+1)!}{(2j+5)!}. \tag{32}$$

and hence

$$b_j = \frac{360(2j+7/2)(j+1)^2(j+5/2)^2(2j)!}{((2j+2)(2j+3)(2j+4)(2j+5) - 360)\,(2j+6)!}. \tag{33}$$

Then we have[5] $\sum_{j=1}^\infty b_j \approx 0.0118141$, $b_0 = -0.045573$, whence

$$\langle L_1^{-1}\phi, \phi\rangle = a\sum_{j=0}^\infty b_j < 0,$$

whence we conclude the stability of the wave (10).

4.2. **On the instability of the wave (11) of the fourth order Klein-Gordon model.** We need to consider the eigenvalue problem (21), with $\mathcal{L}, \mathcal{J}$ given in (23). Based on the index counting theory, (26) and the fact that $n(\mathcal{L}) = 1$ by proposition 2, we are interested in the number of negative eigenvalues of the matrix

$$D = \begin{pmatrix} \langle \mathcal{L}^{-1}\mathcal{J}^{-1}\phi_1, \mathcal{J}^{-1}\phi_1\rangle & \langle \mathcal{L}^{-1}\mathcal{J}^{-1}\phi_1, \mathcal{J}^{-1}\phi_2\rangle \\ \langle \mathcal{L}^{-1}\mathcal{J}^{-1}\phi_2, \mathcal{J}^{-1}\phi_1\rangle & \langle \mathcal{L}^{-1}\mathcal{J}^{-1}\phi_2, \mathcal{J}^{-1}\phi_2\rangle \end{pmatrix}$$

where the two elements of the kernel are given by

$$\phi_1 = \begin{pmatrix} \phi' \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} 0 \\ \phi \\ 0 \\ 0 \end{pmatrix}$$

Since

$$\mathcal{J}^{-1} = \begin{pmatrix} 0 & 2\beta & -1 & 0 \\ -2\beta & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{34}$$

[5]Here we have used Mathematica for an approximation of the value of the series

We have

$$
\begin{aligned}
D_{11} &= \langle \mathcal{L}^{-1}\mathcal{J}^{-1}\phi_1, \mathcal{J}^{-1}\phi_1\rangle = \\
&= \left\langle \begin{pmatrix} L_1^{-1} & 0 & 0 & 0 \\ 0 & L_2^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2\beta\phi' \\ \phi' \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2\beta\phi' \\ \phi' \\ 0 \end{pmatrix} \right\rangle = \\
&= 4\beta^2 \langle L_2^{-1}\phi', \phi'\rangle + \|\phi'\|^2 > 0,
\end{aligned}
$$

since $L_2^{-1}$ is positive on $Ker[L_2]^{\perp} = span[\phi]^{\perp}$. A quick inspection shows $D_{12} = D_{21} = 0$, while

$$D_{22} = 4\beta^2 \langle L_1^{-1}\phi, \phi\rangle + \|\phi\|^2.$$

Thus, we have reduced matters to the sign of $D_{22}$, as usual. It turns out that $D_{22} > 0$, which in view of (26) implies a real instability, since then $n(D) = 0$, while $n(\mathcal{L}) = 1$, whence the left-hand side of (26) is one. Thus, it remains to show that $D_{22} > 0$. We apply again Albert's theory, see Section B.2 in the Appendix.

We have in fact just evaluated the quantity $\langle L_1^{-1}\phi, \phi\rangle$ in our Schrödinger calculations. With the same $\lambda_{2j}$ and $b_j$ as in (32), (33) respectively, we find

$$\langle L_1^{-1}\phi, \phi\rangle = \frac{1}{3}\left(\sqrt{\frac{9}{10}}\right)^2 \frac{1}{\sqrt{1/20}} \left(\frac{2^3\Gamma(2)}{\pi\Gamma(2)}\right)^2 \left(\sum_{j=0}^{\infty} b_j\right) \approx -0.0979003,$$

However, for the function $\phi$ defined in (10), $\|\phi\|^2 \sim 1.7888543...$ whence

$$D_{22} = 4\beta^2 \langle L_1^{-1}\phi, \phi\rangle + \|\phi\|^2 \sim 0.802019... > 0.$$

**Appendix** A. **The Polya-Szegö inequality for fractional Laplacians.** Here, we present a proof of the Polya-Szegö inequality for fractional Laplacians.

**Proposition 3.** *Let $s \in (0, 1]$, $d \geq 1$. Then, for all functions $u \in \dot{H}^s$, we have that its decreasing rearrangement $u^* \in \dot{H}^s$ and moreover*

$$\|(-\Delta)^{s/2}u\|_{L^2(\mathbf{R}^d)} \geq \|(-\Delta)^{s/2}u^*\|_{L^2(\mathbf{R}^d)}. \tag{35}$$

*In addition, equality is achieved if and only if there exists $x_0 \in \mathbf{R}^d$ and a decreasing function $\rho : \mathbf{R}_+ \to \mathbf{R}_+$, so that $u(x) = \rho(|x - x_0|)$.*

**Note:** The classical Polya-Szegö inequality corresponds to the particular case $s = 1$.

*Proof.* Let $s < 1$ and define

$$c_s := \int_0^{\infty} \frac{1 - e^{-y}}{y^{1+s}} dy.$$

Setting $y = 4\pi^2|\xi|^2 t$, we have the representation

$$(2\pi|\xi|)^{2s} = \frac{1}{c_s}\int_0^{\infty} \frac{1 - e^{-4\pi^2 t|\xi|^2}}{t^{1+s}} dt.$$

Equivalently

$$(-\Delta)^s = \frac{1}{c_s}\int_0^{\infty} \frac{1 - e^{t\Delta}}{t^{1+s}} dt.$$

Since $e^{t\Delta} f = K_t * f$ and $K_t(x) = (4\pi t)^{-d/2} e^{-|x|^2/(4t)}$ is strictly symmetric decreasing, we have by the simple rearrangement inequality $\int fg \leq \int f^* g^* dx$ (see Theorem 3.4, [18]) that

$$\langle e^{t\Delta} u, u\rangle = \langle K_t * u, u\rangle \leq \langle K_t * u^*, u^*\rangle = \langle e^{t\Delta} u^*, u^*\rangle.$$

and equality is achieved only if $u(x) = \rho(|x - x_0|)$ for a decreasing function $\rho : \mathbf{R}_+ \to \mathbf{R}_+$ and $x_0 \in \mathbf{R}^d$. Thus,

$$\|(-\Delta)^{s/2} u\|^2 = \langle (-\Delta)^s u, u\rangle = \frac{1}{c_\beta} \int \frac{\langle u, u\rangle - \langle e^{t\Delta} u, u\rangle}{t^{1+\beta}} dt \geq$$

$$\geq \quad \frac{1}{c_\beta} \int \frac{\langle u^*, u^*\rangle - \langle e^{t\Delta} u^*, u^*\rangle}{t^{1+\beta}} dt = \langle (-\Delta)^s u^*, u^*\rangle = \|(-\Delta)^{s/2} u^*\|^2.$$

Moreover, equality is possible only if $u(x) = \rho(|x - x_0|)$, as explained above. $\square$

**Appendix** B. **Some aspects of Albert's total positivity theory.** In this section, we present some basic results from John Albert's positivity theory, [2]. The goal is to describe the basic structure of the spectrum of the self-adjoint operators $L_1, L_2$ introduced in (19), as well as the computations of certain Vakhitov-Kolokolov type quantities as required in the consideration of the eigenvalue problems (18) and (21). Most of the results stated here are due to Albert, [2], so we state them without proofs by referring to the original paper. Back to the specifics, let $\mathcal{T}$ be defined by

$$\mathcal{T} g(x) = M g(x) + \omega g(x) - \varphi^p(x) g(x),$$

where $p \geq 1$ is an integer, $\omega$ is a real parameter, $\varphi$ is real-valued solution of

$$(M + \omega)\varphi = \frac{1}{p+1} \varphi^{p+1}.$$

having a suitable decay at infinity, and $M$ is defined as a Fourier multiplier operator by $\widehat{Mg}(\xi) = m(\xi)\hat{g}(\xi)$, where $m(\xi)$ is a measurable, locally bounded, even function on $\mathbb{R}$ satisfying

- $m(\xi) \sim |\xi|^\mu$ for $|\xi| >> 1$
- $m(\xi) > b$,

where $A_1$ and $A_2$ are positive real constants, $\mu \geq 1$, and $\xi_0$ and $b$ are real numbers. Under these assumptions, $\mathcal{T}$ is self-adjoint, with a.c. spectrum consisting of $[\omega, \infty)$ and (at most) finitely many, counting multiplicities, eigenvalues in $(-\infty, \omega]$.

In order to obtain additional spectral properties of $\mathcal{T}$, Albert, [2] introduces a one-parameter family of operators $\{S_\theta\}_{\theta \geq 0}$, on $L^2(\mathbb{R})$ by

$$S_\theta g(x) = \frac{1}{\omega_\theta(x)} \int_{\mathbb{R}} K(x - y) g(y)\ dy,$$

where $K(x) = \widehat{\varphi^p}(x)$ and $\omega_\theta(x) = m(x) + \theta + \omega$, so that the operators $S_\theta$ act on the Hilbert space

$$X = \{g \in L^2(\mathbb{R}); \|g\|_{X,\theta} = \left(\int_{\mathbb{R}} |g(x)|^2 \omega_\theta(x)\ dx\right)^{1/2} < \infty\}.$$

It is not hard to see that $\{S_\theta\}_{\theta > 0}$ are compact symmetric operators on $X$, whence one gets

**Corollary 1.** *$-\theta$ is an eigenvalue of $\mathcal{T}$ (as an operator acting on $L^2(\mathbb{R})$) with eigenfunction $g$ if and only if , 1 is an eigenvalue of $S_\theta$ (as an operator on $X$) with eigenfunction $\hat{g}$. In particular, both eigenvalues have the same multiplicity. In addition, $S_\theta$ has a family of eigenvectors $\{\psi_i(\theta)\}_{i=0}^{\infty}$ forming an orthonormal basis of $X$. Moreover, the corresponding eigenvalues $\{\lambda_i(\theta)\}_{i=0}^{\infty}$ are real and can be ordered as follows*

$$|\lambda_0(\theta)| \geq |\lambda_1(\theta)| \geq \cdots \geq 0.$$

There is also a result of Krein-Rutman-type, namely

**Lemma 1.** *The eigenvalue $\lambda_0(0)$ of $S_0$ is positive, simple, and has a strictly positive eigenfunction $\psi_{0,0}(x)$. Moreover, $\lambda_0(0) > |\lambda_1(0)|$.*

The main result in Albert's theory is due to fact that the kernel $K = \widehat{\varphi^p}$ lies in a class of functions, which we describe below: a function $h : \mathbb{R} \to \mathbb{R}$ is said to be in the class $PF(2)$ if :

(1) $h(x) > 0$ for all $x \in \mathbb{R}$;
(2) for any $x_1, x_2, y_1, y_2 : x_1 < x_2, y_1 < y_2$, there is

$$h(x_1 - y_1)h(x_2 - y_2) - h(x_1 - y_2)h(x_2 - y_1) \geq 0$$

with a strict inequality whenever the intervals $(x_1, x_2)$ and $(y_1, y_2)$ do intersect.

The central result of [2] is then summarized in

**Theorem 3.** *Suppose $\hat{\varphi} > 0$ on $\mathbb{R}$ and $K = \widehat{\varphi^p} \in PF(2)$. Then $\mathcal{T}$ has a simple, negative eigenvalue $\kappa$, and the eigenvalue 0 of $\mathcal{T}$ is simple.*

With this, we are now ready to present the proof of Proposition 2.

B.1. **Proof of Proposition 2.** The statement for $L_1$ has been already established in [2]. Regarding $L_2$, the theory developed in the last paragraphs cannot be applied directly to $L_2$. On the other hand, to $L_2$, we can still be associated to the family of operators $S_\theta, \theta \geq 0$. It is clear that $L_2\phi = 0$. By Corollary 1, we obtain that 1 is an eigenvalue of $S_0$ with eigenfunction $\hat{\phi}$.

We now claim that if $\lambda_0(0)$ is the first eigenvalue of $S_0$, we have $\lambda_0(0) = 1$. The proof is by contradiction. Assume that $\lambda_0(0) \neq 1$ and let $\psi_{0,0}$ be the associated eigenfunction. By Lemma 1.9 we have $\psi_{0,0} > 0$. Since

$$(\widehat{sech^2(\cdot)})(\xi) = \frac{\pi\xi}{\sinh(\frac{\pi\xi}{2})} > 0$$

it follows that $\hat{\phi} > 0$, hence the scalar product in $L^2(\mathbb{R})$ between $\psi_{0,0}$ and $\hat{\phi}$ is strictly positive, which contradicts the fact that the eigenfunctions are orthogonal. Thus 1 is a simple eigenvalue of $S_0$, then zero is a simple eigenvalue of $L_2$.

It remains to show that $L_2$ has no negative eigenvalues. To do so, it is sufficient to prove that 1 is not en eigenvalue of $S_\theta$ for any $\theta > 0$. We know that

$$\lim_{\theta \to \infty} \lambda_0(\theta) = 0$$

and $\theta \in [0, \infty) \mapsto \lambda_0(\theta)$ is a strictly decreasing function. Thus for $\theta > 0$ and $i \geq 1$,

$$|\lambda_i(\theta)| \leq \lambda_0(\theta) < \lambda_0(0) = 1.$$

Thus it implies that 1 cannot be an eigenvalue of $S_\theta, \theta > 0$, hence $L_1$ cannot allow negative eigenvalues.

B.2. **Computing the Vakhitov-Kolokolov type quantities for $sech^r$ solutions using Albert's approach.** For $\varphi(x) = (sech(x))^r, r = \frac{2n}{p}$, it was established that (see Lemma 4.7, [2]) there exist unique $(n+1)$ tuple $a_0, \ldots, a_n$, so that

$$\sum_{i=0}^{n} a_i(\partial^{2i}\varphi) = \frac{\varphi^{p+1}}{p+1}.$$

Thus, upon introducing the differential operator $M_{n,p} := \sum_{i=1}^{n} \partial^{2i}$, and denoting $C_{n,p} := a_0$, we see that $\varphi$ satisfies the profile equation

$$(M_{n,p} + a_0)\varphi = \frac{\varphi^{p+1}}{p+1}. \tag{36}$$

With this notations, Albert has shown (see Theorem 4.10 in [2]) the following formula

$$\langle (M_{n,p} + C_{n,p} - \varphi^p)^{-1}\varphi, \varphi\rangle = a\sum_{j=0}^{\infty} b_j \tag{37}$$

where $a = \left(\frac{2^{n+r-1}\Gamma(r)}{\pi\Gamma(n)}\right)^2 > 0$, $\lambda_m = \frac{\Gamma(r+m)}{\Gamma(r+1)}\frac{\Gamma(r+2n+1)}{\Gamma(r+2n+m)}$ and

$$b_j = \left(\frac{\lambda_{2j}}{1-\lambda_{2j}}\right)\left\{\frac{\Gamma(2j+1)\cdot(2j+n+r-\frac{1}{2})}{\Gamma(2j+2n+2r-1)}\right\}\left\{\frac{\Gamma(j+n)\Gamma(j+n+r-\frac{1}{2})}{\Gamma(j+1)\Gamma(j+r+\frac{1}{2})}\right\}^2.$$

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