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Regularity of ground state solutions of dispersion managed nonlinear schrödinger equations☆

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Abstract

We consider the dispersion managed nonlinear Schrödinger equation (DMNLS) in the case of zero residual dispersion. Using dispersive properties of the equation and estimates in Bourgain spaces we show that the ground state solutions of DMNLS are smooth. The existence of smooth solutions in this case matches the well-known smoothness of the solutions in the case of nonzero residual dispersion. In the case $x \in \mathbb{R}^2$ we prove that the corresponding minimization problem with zero residual dispersion has no solution.

Keywords: Dispersion managed nonlinear Schrödinger equation; Regularity; Ground states

1. Introduction and main result

Our work is motivated by the study of parametrically excited NLS with periodically varying dispersion coefficient

$$iu_t + D(t)u_{xx} + C(t)|u|^2u = 0,$$

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which arises as an envelope equation in the problem of an electromagnetic wave propagating in an optical waveguide. The balance between the dispersion and the nonlinearity in this equation is the key factor that determines the existence of stable pulses. In the last decade, a technique that uses fibers with alternating sections having opposite dispersion was introduced. This technology, called dispersion management, proved to be incredibly successful in producing stable, soliton-like pulses. The idea is to use rapidly varying dispersion with approximately zero mean and small nonlinearity in hope that the balance between the small residual dispersion and the small nonlinearity will produce a soliton-like solutions. There have been an enormous amount of technological advances in this direction with an array of numerical and phenomenological explanations and a recent theoretical understanding of the strong stability properties of the dispersion managed (DM) systems. The envelope equation that describes the propagation of electromagnetic pulses in optical fibers in the regime of strong dispersion management, derived by Gabitov and Turitsyn in 1996 [6,7] is a nonlinear Schrödinger equation with periodically varying coefficients. After rescaling the equation takes the form

$$iu_t + d(t)u_{xx} + \varepsilon(|u|^2 u + \alpha u_{xx}) = 0, \tag{1}$$

where t is the propagation distance, x is the retarded time and $d(t)$ is the mean-zero component of the dispersion, see [17]. Note that the average dispersion and nonlinearity are small compared to the local dispersion, which is a characteristic feature of the strong dispersion management. Performing Van der Pol transformation in (1) and averaging in the Hamiltonian we obtain the averaged variational principle

$$\langle H \rangle = \varepsilon \int_{-\infty}^{+\infty} \int_0^1 \left(\alpha |v_x|^2 - \frac{1}{2} |T(t)v|^4 \right) dx\, dt \tag{2}$$

with the corresponding Euler–Lagrange equation (averaged), see [1,7]

$$iv_t + \varepsilon \alpha v_{xx} + \varepsilon \langle Q \rangle (v, v, v) = 0, \tag{3}$$

where

$$\langle Q \rangle (v_1, v_2, v_3) = \int_0^1 Q(v_1, v_2, v_3, t)\, dt.$$

Here $T(t)$ is the fundamental solution of $iu_t + d(t)u_{xx} = 0$ and

$$Q(v_1, v_2, v_3, t) = T^{-1}(t)(T(t)v_1 T(t)v_2 \overline{T(t)v_3}).$$

In [17] the existence of ground state solution for the averaged equations is proved, as well as an averaging result, which guarantees the existence of nearly periodic stable pulses, see also [4]. The ground state of the averaged equation exists as a solution of the constrained minimization problem

$$P_\lambda = \inf \left\{ E(v) = \langle H \rangle (v),\, v \in H^1,\, \int_{-\infty}^{+\infty} |v|^2 dx = \lambda \right\}. \tag{4}$$

This result is for the case of positive average dispersion α and using bootstrapping procedure it is shown that the minimizer is smooth in this case. The variational problem in the case of zero-average dispersion is more subtle due to the absence of a priori bounds in spaces different from L^2. In this case the functional is formally the singular perturbation limit $\alpha \to 0$ of (4), see [11,17]. In [10] the corresponding minimization problem

$$P_\lambda = \inf \left\{ \varphi(u),\, u \in L^2,\, \int_{-\infty}^{+\infty} |u|^2\, dx = \lambda \right\}, \tag{5}$$

where $\varphi(u) = -\int_0^1 \int_{-\infty}^{+\infty} |e^{it\partial_x^2} u(x)|^4\, dx\, dt$ has been studied. By $e^{it\partial_x^2}$ we denote the semigroup generated by the free Schrödinger equation in one dimension, i.e. $u(t,x) = (e^{it\partial_x^2} u_0)(x)$ solves

$$iu_t + u_{xx} = 0, \;\; u(0,x) = u_0(x).$$

Exploring the dispersive properties of the Schrödinger evolution and using Lion's concentration compactness in L^2, the existence of a minimizer $u \in L^2 \cap L^\infty$ has been derived.

In the current paper, we follow the same idea as in [10], but make use of Bourgain spaces $X_{s,b}$ to simplify the proof and show that the existing minimizer u is smooth. More precisely, we prove the following theorem.

Theorem 1. *The minimization problem* (5) *has a solution* $u \in C^\infty \cap L^2$.

It is interesting to study the two-dimensional case $x \in \mathbb{R}^2$, which is physically relevant since x is the coordinate of the sections orthogonal to the fiber and t is the distance along the fiber. In this case the corresponding model is the variable coefficients nonlinear Schödinger equation in two-space dimensions

$$iu_t + d(t)\Delta u + c(t)|u|^2 u = 0. \tag{6}$$

The results in [17] transfer to the two-dimensional case. There exists a solution for every $\alpha, \lambda > 0$ of the corresponding variational problem

$$\min \left\{ \alpha \int_{\mathbb{R}^2} |\nabla u|^2\, dx - \frac{1}{2} \int_0^1 \int_{\mathbb{R}^2} |U(t)u|^4\, dx\, dt : \;\; u \in \mathbf{H^1},\, \int_{\mathbb{R}^2} |u|^2 dx = \lambda \right\}. \tag{7}$$

More recently Kunze in [12] has shown that again in the case of nonzero residual dispersion the functional

$$\varphi(u) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + |u|^2)\,dx - \int_0^1\int_{\mathbb{R}^2}|U(t)u|^4\,dx\,dt, \quad u \in \mathbf{H^1}, \tag{8}$$

where $U(t)u_0 = e^{it\Delta}u_0$ is the evolution operator of the free Schrödinger equation admits a sequence $(u_j) \subset \mathbf{H^1}$ of critical points such that u_j are radially symmetric and $|u_j|_{H^1} \to \infty$ as $j \to \infty$. Here $\alpha \neq 0$ is taken equal to 1 without loss of generality and the constraint $\|u\|_{L^2} = 1$ is included in the functional. In [10] the author posed the problem about the existence of a constrained minimizer for the functional

$$\varphi(u) = -\int_0^1\int_{\mathbb{R}^2}|U(t)u|^4\,dx\,dt, \quad u \in L^2 \tag{9}$$

in the two-dimensional case $x \in \mathbb{R}^2$. In the next theorem we give negative answer to this question.

Theorem 2. *In $\mathbb{R}^2$ a solution of the constrained minimization problem*

$$P = \inf\left\{\varphi(u) = -\int_0^1\int_{\mathbb{R}^2}|U(t)u|^4\,dx\,dt,\ u \in L^2,\ \|u\|_{L^2} = 1\right\}$$

does not exist.

Note that the questions above are related with the question of existence of a maximizer and an exact constant in the Strichartz inequality

$$\|u\|_{L^p(R^{n+1})} \leqslant S\|f\|_{L^2(R^n)}, \quad p = 2 + 4/n,$$

whenever $u(t,x)$ is a solution of the equation $i\partial_t u = \Delta u$ with initial data $u(0,x) = f(x)$. In this case the integral in t is over the infinite interval $(0,\infty)$. It has been shown by Kunze [13] that maximizing function exists in the case $n = 1$, $p = 6$. Recently, in [5] Foschi was able to explicitly construct maximizers when the exponent p is an even integer. In the cases of interest for us, $n = 1$ and 2, the exact constants are given as well as the form of the smooth maximizing functions. Note that we show that for the case $n = 2$ with integration in t over the finite interval $(0, 1)$ the maximizer does not exist.

Another case of interest is to consider a one-dimensional NLS with quintic nonlinearity

$$iu_t + u_{xx} + |u|^4 u = 0,$$

which arises if the electromagnetic field is so strong that higher order nonlinearity can not be neglected. If we introduce dispersion management with rapidly varying dispersion the corresponding model is given by

$$iu_t + d(t)u_{xx} + |u|^4 u = 0.$$

In [17] the authors follow the averaging procedure to produce the equation

$$iv_t + \alpha v_{xx} + bQ_5(v, v, v, v, v) = 0,$$

where $Q_5(v_1, v_2, v_3, v_4, v_5, t) = T^{-1}(t)(T(t)v_1 T(t)v_2 \overline{T(t)v_3} T(t)v_4 \overline{T(t)v_5})$ with the averaged Hamiltonian

$$\langle H \rangle = \int_0^1 \int_{-\infty}^{+\infty} \left(\alpha |v_x|^2 - \frac{1}{2} |T(t)v|^6 \right) dx\, dt.$$

A solution $v \in H^1$ of the constrained minimization problem

$$P_\lambda = \inf \left\{ E(v) = \langle H \rangle (v), v \in H^1, \int_{-\infty}^{+\infty} |v|^2\, dx = \lambda \right\}$$

when $\alpha \neq 0$ was found in [17]. We prove the following:

Theorem 3. *In $\mathbb{R}^1$ a solution for the constrained minimization problem*

$$P = \inf \left\{ \varphi(u) = -\int_0^1 \int_{-\infty}^{\infty} |T(t)u|^6\, dx\, dt,\ \ u \in L^2,\ \ \int_{-\infty}^{+\infty} |u|^2\, dx = 1 \right\}$$

does not exist.

2. Proof of Theorem 1

Introduce the Bourgain spaces $X_{s,b}$ [2,3] as the set of all functions u with

$$\int |\hat{u}(\xi, \tau)|^2 \langle \tau - |\xi|^2 \rangle^{2b} \langle \xi \rangle^{2s}\, d\xi\, d\tau < \infty,$$

where $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ and $\langle \tau - |\xi|^2 \rangle := (1 + |\tau - |\xi|^2|^2)^{1/2}$ and $\hat{u}(\xi, \tau)$ is the time–space Fourier transform. We also introduce the space $X^-_{s,b}$ as

$$X^-_{s,b} := \left\{ u : \int |\hat{u}(\xi, \tau)|^2 \langle \tau + |\xi|^2 \rangle^{2b} \langle \xi \rangle^{2s}\, d\xi\, d\tau < \infty \right\}.$$

Note that $X_{s,b}$ spaces are Hilbert spaces with norm

$$\|u\|_{X_{s,b}} = \int |\hat{u}(\xi,\tau)|^2 \langle \tau - |\xi|^2\rangle^{2b} \langle \xi\rangle^{2s}\, d\xi\, d\tau$$

and that $\|u\|_{X_{s,b}} = \sup_{v\in X^-_{-s,-b}} \int u\overline{v}\, dx\, dt$. Thus $v \in X^-_{s,b}$ if and only if $\overline{v} \in X_{s,b}$. We include the following well-known lemma for convenience next.

Lemma 4. *Let* $\psi \in C_0^\infty(\mathbb{R}^1)$, $\operatorname{supp}\psi \subset (-1,1)$. *Then*,
(1) $\left\|\psi(t)e^{it\partial_x^2}u_0\right\|_{X_{s,b}} \leqslant C_b\|u_0\|_{H^s}$;
(2) $\|u\|_{L_t^\infty H_x^s} \leqslant C_\varepsilon \|u\|_{X_{s,1/2+\varepsilon}}$.

Proof. To prove (1), compute the Fourier transform of the left-hand side

$$\mathcal{F}(\psi(t)e^{it\partial_x^2}u_0)(\tau,\xi) = \hat{\psi}(\tau - |\xi|^2)\hat{u_0}(\xi).$$

Thus

$$\begin{aligned}&\left\|\psi(t)e^{it\partial_x^2}u_0\right\|_{X_{s,b}}\\ &\quad\leqslant \int |\hat{\psi}(\tau-|\xi|^2)|^2|\hat{u_0}(\xi)|^2\langle\tau-|\xi|^2\rangle^{2b}\langle\xi\rangle^{2s}\, d\xi\, d\tau.\end{aligned}$$

Then (1) follows from $\int |\hat{\psi}(\tau-|\xi|^2)|^2\langle\tau-|\xi|^2\rangle^{2b}\, d\tau \leqslant C_b$.
For part (2) since $\|u\|_{L_t^\infty L_x^2} \leqslant \|\hat{u}\|_{L_\xi^2 L_\tau^1}$ we have

$$\begin{aligned}\|u\|^2_{L_t^\infty H_x^s} &\leqslant \int\left(\int |\hat{u}(\tau,\xi)|\, d\tau\right)^2 \langle\xi\rangle^{2s}\, d\xi\\ &\leqslant \int\left(\int |\hat{u}(\tau,\xi)|^2\langle\tau-|\xi|^2\rangle^{1+2\varepsilon}\, d\tau\right)\cdot\left(\int \frac{d\tau}{\langle\tau-|\xi|^2\rangle^{1+2\varepsilon}}\right)\langle\xi\rangle^{2s}\, d\xi\\ &\leqslant C_\varepsilon\|u\|^2_{X_{s,1/2+\varepsilon}}. \qquad \square\end{aligned}$$

We will need to use the following lemma [15, p. 21] on the smoothing effect of the Duhamel operator on the space $X_{s,b}$, see also [14].

Lemma 5. *Let* ψ *be a smooth characteristic function of the interval* $[-1,1]$. *Then for any* $\varepsilon > 0$

$$\left\|\psi(s)\int_0^s e^{i(s-t)\partial_x^2}F(t)\, dt\right\|_{X^{s,1/2+\varepsilon}} \leqslant \|F\|_{X^{s,-1/2+2\varepsilon}}.$$

Next, we introduce the Littlewood–Paley decomposition. Let $\varphi \in C_0^\infty(\mathbf{R^1})$ and $\varphi(\xi) = 1$ if $|\xi| \leqslant 1$ and $\varphi(\xi) = 0$ if $|\xi| > 2$. Define the function $\psi(\zeta) = \varphi(\zeta) - \varphi(2\zeta)$. Then

$$\varphi(\zeta) + \sum_{k=1}^{\infty} \psi(2^{-k}\zeta) = 1$$

for every $\zeta \in \mathbf{R}$, $\boldsymbol{\zeta} \neq \mathbf{0}$. Define the Littlewood–Paley operators as

$$\widehat{P_k f}(\zeta) = \psi(2^{-k}\zeta)\hat{f}(\zeta)$$

and

$$\widehat{P_0 f}(\zeta) = \varphi(\zeta)\hat{f}(\zeta) \sim \chi_{[-1,1]}(\zeta)\hat{f}(\zeta).$$

Note that $\widehat{P_k f}(\zeta) \neq 0$ only if $2^{k-1} \leqslant |\zeta| \leqslant 2^{k+1}$.

Let $P_{k-5<.<k+5}$ be the operator

$$P_{k-5<.<k+5} = \sum_{i=-5}^{i=5} P_{k+i}.$$

For simplicity we will denote $u_k = P_k u$ and $u_{k-5<.<k+5} = P_{k-5<.<k+5}u$ from now on.

We will use the following main theorem, the proof of which will be given in the next section. In this theorem and in what follows, L^2-norms will refer to spatial L^2_x-norms unless specifically stated otherwise.

Theorem 6. *For every* $l > 0$

$$\|P_l(\langle Q\rangle(u))\|_{L^2} \lesssim C(2^{-l(1/2-10\varepsilon)}\|u\|^3_{L^2} + \|u_{l-5<.<l+5}\|^3_{L^2})$$

with C *independent of* l *and small* $\varepsilon > 0$.

Remark. The estimate in Theorem 6 can be improved to

$$\|P_l(\langle Q\rangle(u))\|_{L^2} \lesssim C(2^{-l(1/2-10\varepsilon)}\|u_{>l-2}\|_{L^2}\|u\|^2_{L^2} + \|u_{l-5<.<l+5}\|^3_{L^2}).$$

We will postpone the proof of this theorem and discuss the minimization problem instead. We want to minimize

$$\varphi(u) = -\int_0^1 \int_{-\infty}^{+\infty} |e^{it\partial_x^2}u(x)|^4\,dx\,dt$$

subject to $\|u\|_{L^2} = \lambda$. But if we choose $v(x) = \frac{u(x)}{\sqrt{\lambda}}$ then $\|v(x)\|_{L^2} = 1$ and $\varphi(u) = \varphi(\sqrt{\lambda}v) = \lambda^2\varphi(v)$ we see that it is enough to consider the minimization problem

$$\mathcal{P}_1 = \inf\{\varphi(u) : u \in L^2,\ \ \|u\|_{L^2} = 1\}.$$

By Ekeland's principle, we can choose the minimizing sequence $\{u^m\}$ in such a way that

$$\mathcal{P}_1 u^m + \langle Q\rangle(u^m) \to 0 \text{ in } L^2.$$

Thus we have the following problem:

$$\begin{aligned} &\left\|u^m\right\|_{L^2} = 1,\\ &\varphi(u^m) \to \inf_{\|u^m\|_{L^2}=1} \varphi(u),\\ &g^m = \mathcal{P}_1 u^m + \langle Q\rangle(u^m) \to 0. \end{aligned} \tag{10}$$

Definition 7. Fix $\delta > 0$ and $\{u^m\}$ with $\|u^m\|_{L^2} = 1$. We say that l is an exceptional frequency for $\{u^m\}$ if $\left\|u^m_{l-5<.<l+5}\right\|_{L^2} \geqslant \delta$ for all m.

Proposition 8. *There exist finitely many exceptional frequencies. Also, there exists a finite set A of frequencies and a subsequence such that whenever $l \notin A$ there exists $m = m(l)$ such that $\left\|u^m_{l-5<.<l+5}\right\|_{L^2} \leqslant \delta$ for all $m \geqslant m(l)$.*

Proof. It is clear by the definition that the number of exceptional frequencies cannot exceed $\dfrac{10}{\delta^2}$. To construct the set A and the corresponding subsequence, take all the exceptional frequencies for $\{u^m\}$ and call that set A. If $l \notin A$ there exists an infinite subsequence $\{u^{m_k}\}$ such that $\left\|u^{m_k}_{l-5<.<l+5}\right\|_{L^2} < \delta$. To this subsequence apply the same procedure for the next $l' \notin A$, etc. In the end, take the diagonal subsequence which will satisfy the condition. □

Consider now the set $\mathbf{N} \setminus A$ of frequencies. We have that for every $l \in \mathbf{N} \setminus A$ and every $m > m(l)$

$$\begin{aligned} \mathcal{P}_1\left\|u^m_l\right\|_{L^2} - \left\|g^m_l\right\|_{L^2} &\leqslant \left\|\mathcal{P}_1 u^m_l - g^m_l\right\|_{L^2} = \left\|P_l(\langle Q\rangle(u^m))\right\|_{L^2}\\ &\leqslant C2^{-l(1/2-10\varepsilon)}\left\|u^m\right\|^3_{L^2} + C\left\|u^m_{l-5<.<l+5}\right\|^3_{L^2}\\ &\leqslant C2^{-l(1/2-10\varepsilon)} + C\delta^2\left(\sum_{i=-5}^{5}\left\|u^m_{l-i}\right\|_{L^2}\right). \end{aligned}$$

Dividing through by $\mathcal{P}_1$ and renaming the constants gives us

$$\left\|u_l^m\right\|_{L^2} - c\left\|g_l^m\right\|_{L^2} \leqslant C2^{-l(1/2-10\varepsilon)} + C\delta^2 \left(\sum_{i=-5}^{5} \left\|u_{l-i}^m\right\|_{L^2}\right).$$

Recall now that $\left\|g_l^m\right\|_{L^2} \to 0$ and that by the result of [10], we have $u_l^m \to u_l$ in L^2. Taking limit in m in the last inequality yields

$$\|u_l\|_{L^2} \leqslant C2^{-l(1/2-10\varepsilon)} + C\delta^2 \left(\sum_{i=-5}^{5} \|u_{l-i}\|_{L^2}\right)$$

for every $l \notin A$. Taking sufficiently big constant C will ensure that these inequality will remain true for $l \in A$, since that set is of finite cardinality.
Let $a_l = \|u_l\|_{L^2}$. In terms of a_l the last estimate reads

$$a_l \leqslant C2^{-l(1/2-10\varepsilon)} + C\delta^2(a_{l-5} + \cdots + a_{l+5}),$$

We have the following lemma.

Lemma 9. *Let $\sigma > 0$. Then there exists a constant $\kappa_0 = \kappa_0(\sigma)$, so that whenever $0 < \kappa < \kappa_0$, the sequence $a_l \in l^2$ and $0 < a_l \leqslant C2^{-l\sigma} + \kappa(a_{l-5} + \cdots + a_{l+5})$, for all positive integers l, one has*

$$a_k \leqslant C_{\varepsilon,\sigma}(1 + \|\{a_l\}\|_{l^2})2^{-k\sigma}$$

for all $k > 0$.

We include the proof of the lemma in the appendix. Assuming its validity, we get by choosing an appropriate small $\delta > 0$ such that $\|u_l\|_{L^2} \leqslant C2^{-l(1/2-11\varepsilon)}$. For sufficiently small ε we have the estimate $\|u_l\|_{L^2} \leqslant C2^{-l/3}$. According to the Remark after Theorem 6 we have

$$\|u_l\|_{L^2} = \|P_l(\langle Q\rangle(u))\|_{L^2} \leqslant C2^{-l/3}2^{-l/3} + C(2^{-l/3})^3 \leqslant C2^{-2l/3}.$$

This gives already $u \in H^{2/3-}$ and shows that by iteration one can prove that the solution u is actually smooth, i.e. $u \in C^\infty$.

Interestingly, to prove that $u^m \to u$ in L^2, Kunze has shown that the only possible case is when the sequence $\{\hat{u}^m\}$ is tight. Using our arguments, we are in fact showing something more, namely

$$\int_{|\xi|\geqslant R} |\hat{u}^m(\xi)|^2 \leqslant \sum_{l:2^l\geqslant R} \|u_l^m\|_{L^2}^2 \lesssim \sum_{l:2^l\geqslant R} \frac{1}{2^{2l(1/2-10\varepsilon)}} \lesssim \frac{1}{R^{1-20\varepsilon}}.$$

which implies the tightness of $\{\hat{u}^m\}$.

3. Proof of Theorem 6

In this section, we will give the proof of Theorem 6. To do this, we need to introduce a dyadic decomposition in the "variable" $\tau - |\xi|^2$, i.e.

$$\hat{u}(\tau,\xi) = \sum_{j=0}^{\infty} \psi(2^{-j}(\tau - |\xi|^2))\widehat{u}(\tau,\xi) + \varphi(2(\tau - |\xi|^2))\widehat{u}(\tau,\xi)$$

and denote

$$\widehat{\Pi_j u(\tau,}\xi) = \psi(2^{-j}(\tau - |\xi|^2))\hat{u}(\tau,\xi) = \widehat{u^j}$$

and

$$\widehat{\Pi_0 u(\tau,}\xi) = \varphi(2(\tau - |\xi|^2))\hat{u}(\tau,\xi) = \widehat{u^0}.$$

Then

$$\|u\|_{X^{0,b}} \sim \left(\sum_{j=0}^{\infty} 2^{2jb}\|\Pi_j(u)\|_{L^2}^2\right)^{1/2}$$

for $b > \frac{1}{2}$.

Next, we estimate the norm of the projection P_l of the quantity

$$\langle Q\rangle(u) = \int_0^1 T^{-1}(t)(T(t)uT(t)u\overline{T(t)u})\,dt$$

in the Sobolev space H^s. Take a smooth cutoff function $\varphi(q)$ adapted to the interval (0,1) (following an idea of Kunze). Then

$$\begin{aligned}\|P_l(\langle Q\rangle(u))\|_{H^s} &= \left\|P_l\left(\int_0^1 e^{-it\partial_x^2}(|e^{it\partial_x^2}u|^2 e^{it\partial_x^2}u)\,dt\right)\right\|_{H^s}\\ &\leqslant \sup_{0\leqslant q\leqslant 1}\left\|\int_0^q e^{i(q-t)\partial_x^2}P_l(|e^{it\partial_x^2}u|^2 e^{it\partial_x^2}u)\,dt\right\|_{H^s}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\| \varphi(q) \int_0^q e^{i(q-t)\partial_x^2} P_l(|e^{it\partial_x^2} u|^2 e^{it\partial_x^2} u)\, dt \right\|_{L_t^\infty H_x^s} \\
&\leqslant \left\| \varphi(q) \int_0^q e^{i(q-t)\partial_x^2} P_l(|e^{it\partial_x^2} u|^2 e^{it\partial_x^2} u)\, dt \right\|_{X^{s,1/2+\varepsilon}}.
\end{aligned}
$$

Lemma 5 implies that

$$
\| P_l(\langle Q\rangle(u))\|_{L^2} \leqslant \left\| P_l(|e^{it\partial_x^2} u|^2 e^{it\partial_x^2} u) \right\|_{X^{0,-1/2+2\varepsilon}}.
$$

Thus we need to estimate in the space $X^{0,-1/2+2\varepsilon}$ and we will use different estimates in the case when all the frequencies are almost the same (harder) and in the case when the frequencies are different. We do this according to the following lemma.

Lemma 10. *Let* $0 < \varepsilon$ *be a sufficiently small number* ($\varepsilon < 1/20$ *will do*). *Let* u, v, w *be sufficiently smooth* (*test*) *functions. Then*
If any two frequencies do not match (*that is* $\max(i, j, k, l) - \min(i, j, k, l) > 5$), *we have*

$$
\begin{aligned}
&\sum_{\max(i,j,k,l)-\min(i,j,k,l)>5} \left\| P_l(u_i \bar{v}_j w_k) \right\|_{X^{0,-1/2+2\varepsilon}} \\
&\lesssim 2^{-l(1/2-\varepsilon)} \|u\|_{X^{0,1/2+\varepsilon}} \|v\|_{X^{0,1/2+\varepsilon}} \|w\|_{X^{0,1/2+\varepsilon}}.
\end{aligned}
$$

In the case, when all frequencies are almost the same ($i, j, k \sim l$),

$$
\begin{aligned}
&\left\| \sum_{(i,j,k):\ \max(i,j,k,l)-\min(i,j,k,l)\leqslant 5} P_l(u_i \bar{v}_j w_k) \right\|_{X^{0,-1/2+2\varepsilon}} \\
&\lesssim \left\| u_{l-5\leqslant \cdot < l+5} \right\|_{X^{0,1/2+\varepsilon}} \left\| v_{l-5\leqslant \cdot < l+5} \right\|_{X^{0,1/2+\varepsilon}} \\
&\quad \times \left\| w_{l-5\leqslant \cdot < l+5} \right\|_{X^{0,1/2+\varepsilon}}.
\end{aligned}
$$

In both cases, the sum is over all nontrivial frequencies, that is $\min(i, j, k, l) \geqslant 0$.

We postpone the proof of Lemma 10 for the appendix in order to finish the proof of Theorem 6.

$$
\| P_l(\langle Q\rangle(u))\|_{L^2} \leqslant \left\| P_l(|e^{it\partial_x^2} u|^2 e^{it\partial_x^2} u) \right\|_{X^{0,-1/2+2\varepsilon}}.
$$

Denote $\tilde{u} = e^{it\partial_x^2} u$ and use Lemma 10 to get

$$\begin{aligned}\left\| P_l(|\tilde{u}|^2\tilde{u})\right\|_{X^{0,-1/2+2\varepsilon}} &\leqslant \sum_{\max(i,j,k,l)-\min(i,j,k,l)>5} \left\| P_l(\tilde{u}_i\bar{\tilde{u}}_j)\tilde{u}_k\right\|_{X^{0,-1/2+2\varepsilon}} \\ &\quad + \left\| \sum_{\max(i,j,k,l)-\min(i,j,k,l)\leqslant 5} P_l(\tilde{u}_i\bar{\tilde{u}}_j\tilde{u}_k)\right\|_{X^{0,-1/2+2\varepsilon}} \\ &\lesssim 2^{-l(1/2-\varepsilon)}\|\tilde{u}\|^3_{X^{0,1/2+\varepsilon}} + \left\|\tilde{u}_{l-5\leqslant\cdot<l+5}\right\|^3_{X^{0,1/2+\varepsilon}} \\ &\lesssim 2^{-l(1/2-\varepsilon)}\|u\|^3_{L^2} + \left\|u_{l-5\leqslant\cdot<l+5}\right\|^3_{L^2},\end{aligned}$$

which is Theorem 6. In the last inequality, we have used Lemmas 4 and 10.

Note that in the sums above $\max(i, j, k) \geqslant l-2$ and hence we have the improved estimate

$$\|P_l(\langle Q\rangle(u))\|_{L^2} \lesssim C(2^{-l(1/2-10\varepsilon)}\|u_{>l-2}\|_{L^2}\|u\|^2_{L^2} + \|u_{l-5<.<l+5}\|^3_{L^2}).$$

4. Two-dimensional dispersion managed nonlinear Schrödinger equation (DMNLS)

In this section, we will give the short proof of Theorem 2. First, denote

$$I(T,\varphi) = \left(\int_0^T \int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4 \, dx\, dt\right)^{1/4}$$

and $C(T) = \sup_{\|\varphi\|_{L^2}=1} I(T,\varphi)$. We will show that $C(1)$ is not achieved, which is equivalent to the statement of Theorem 2.

The Strichartz estimate

$$\|e^{it\Delta}\varphi\|_{L^4} = \left(\int_0^\infty \int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4 \, dx\, dt\right)^{1/4} \leqslant c\|\varphi\|_{L^2}$$

for some constant $c > 0$ gives that

$$C(\infty) < \infty.$$

It is clear also that $C(T)$ is an increasing function of T and $C(T) \leqslant C(+\infty)$. Next, we will show that

$$\lim_{T\to+\infty} C(T) = C(\infty).$$

Indeed, there exists for each $\varepsilon > 0$ a function $\varphi \in L^2$, $\|\varphi\|_{L^2} = 1$ such that $I(\infty, \varphi) > C(\infty) - \varepsilon$. For that φ there exists $T_0 = T_0(\varphi)$ such that for all $T > T_0$ we have

$$\begin{aligned} C(T) &\geqslant \left(\int_0^T \int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4\,dx\,dt\right)^{1/4} \\ &\geqslant \left(\int_0^\infty \int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4\,dx\,dt\right)^{1/4} - \varepsilon \geqslant C(\infty) - 2\varepsilon. \end{aligned}$$

Thus $\lim_{t\to\infty} C(T) = C(\infty)$.

Lemma 11. *For every $T > 0$ we have that $C(T) = C(1)$.*

Proof. The functional $I(T, \varphi)$ scales as follows:

$$I(T, \varphi) = I(1, \sqrt{T}\varphi(\sqrt{T}.)).$$

Thus

$$C(T) = \sup_{\|\varphi\|_{L^2}=1} I(T, \varphi) = \sup_{\|\varphi\|_{L^2}=1} I(1, \sqrt{T}\varphi(\sqrt{T}.)) = \sup_{\|\psi\|_{L^2}=1} I(1, \psi) = C(1)$$

since $\|\psi\|_{L^2} = \|\varphi\|_{L^2} = 1$. $\square$

Suppose now that there exists function φ such that

$$I(1, \varphi) = \left(\int_0^1 \int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4\,dx\,dt\right)^{1/4} = C(1).$$

Then

$$\begin{aligned} C^4(\infty) &\geqslant I^4(\infty, \varphi) = \int_0^\infty \int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4\,dx\,dt \\ &\geqslant \int_0^1 \int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4\,dx\,dt + \int_1^\infty \int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4\,dx\,dt \\ &= C^4(1) + \int_1^\infty \int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4\,dx\,dt. \end{aligned}$$

Thus $\int_1^\infty \int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4\,dx\,dt = 0$ and $\int_{\mathbb{R}^2} |e^{it\Delta}\varphi|^4\,dx = 0$ for almost every t. There exists t_0 such that for every ball $B(0, R) \in \mathbb{R}^2$ $\int_{B(0,R)} |e^{it_0\Delta}\varphi|^4\,dx = 0$. Then by

Hölder we have that for all $R > 0$

$$\int_{B(0,R)} |e^{it_0\Delta}\varphi|^2 dx \leqslant \left(\int_{B(0,R)} |e^{it_0\Delta}\varphi|^4\, dx\right)^{1/2} \left(\int_{B(0,R)} dx\right)^{1/2} = 0.$$

Thus $\int_{\mathbb{R}^2} |e^{it_0\Delta}\varphi|^2\, dx = \int_{\mathbb{R}^2} |\varphi|^2\, dx = 0$, which is a contradiction with $\|\varphi\|_{L^2} = 1$.

5. One-dimensional quintic DMNLS

We will prove Theorem 3 here. As before, denote

$$I(T,\varphi) = \left(\int_0^T \int_{-\infty}^{\infty} |e^{it\partial_x^2}\varphi|^6\, dx\, dt\right)^{1/6}$$

and $C(T) = \sup\limits_{\|\varphi\|_{L^2}=1} I(T,\varphi)$.

$$C(\infty) < \infty$$

is given again by the Strichartz estimate in one dimension,

$$\|e^{it\partial_x^2}\varphi\|_{L^6} = \left(\int_0^{\infty} \int_{-\infty}^{\infty} |e^{it\partial_x^2}\varphi|^6\, dx\, dt\right)^{1/4} \leqslant c\|\varphi\|_{L^2}.$$

We have that $C(T)$ is an increasing function of T with

$$\lim_{T\to+\infty} C(T) = C(\infty).$$

Using the same argument with different scaling

$$I(T,\varphi) = I(1, T^{1/4}\varphi(\sqrt{T}.))$$

we can show that $C(T) = C(1)$ for every $T > 0$. Now if we assume that there exists function φ such that

$$I(1,\varphi) = \left(\int_0^1 \int_{-\infty}^{\infty} |e^{it\partial_x^2}\varphi|^6\, dx\, dt\right)^{1/6} = C(1),$$

we will get a contradiction with $\|\varphi\|_{L^2} = 1$ as above.

Appendix

The proof of Lemma 9 is rather standard and should be available in some form in the literature, but since the author is unaware of such reference, we include it for completeness.

Proof of Lemma 9. For convenience let $a_l = 0$ for all $l \leqslant 0$. For fixed positive integer k, sum both sides in $l \geqslant k$ to get

$$d_k := \left(\sum_{l \geqslant k} |a_l|^2\right)^{1/2} \leqslant C_\sigma 2^{-k\sigma} + C\kappa \left(\sum_{l \geqslant k-5} |a_l|^2\right)^{1/2} \leqslant C_\sigma 2^{-k\sigma} + C\kappa d_{k-5}.$$

This is a well-defined sequence, since $\{a_l\} \in l^2$. Iterate the inequality above to get

$$\begin{aligned} d_k &\leqslant C_\sigma 2^{-k\sigma} + C\kappa(C_\sigma 2^{-(k-5)\sigma} + C\kappa d_{k-5}) \leqslant \cdots \\ &\leqslant C_\sigma \left(\sum_{s=0}^{\infty} (C\kappa 2^{5\sigma})^s\right) 2^{-k\sigma} + (C\kappa)^{[k/5]}. \max(d_0, \ldots, d_5). \end{aligned}$$

The sum in s is estimated by $(1 - C\kappa 2^{5\sigma})^{-1}$, provided $C\kappa 2^{5\sigma} < C\kappa_0 2^{5\sigma} \leqslant 1$, which we require. We also require that $(Ck)^{1/5} \leqslant (Ck_0)^{1/5} \leqslant 2^{-\sigma}$. Finally, observe that $d_k \leqslant \|\{a_l\}\|_{l^2}$. It follows that

$$|a_k| \leqslant d_k \leqslant C_{\sigma,\kappa}(1 + \|\{a_l\}\|_{l^2}) 2^{-k\sigma}. \qquad \square$$

Next, we will show Lemma 10.

Proof of Lemma 10. For the proof of Lemma 10, we rely on the following bilinear estimates of Tao. Namely, in the case of one spatial dimension, it is proved in Proposition 11.1 in [16] that

$$\begin{aligned} &\left\|\Pi_{L_1}(u_i)\Pi_{L_2}(v_j)\right\|_{L^2} \\ &\lesssim \frac{L_1^{1/2} \min(L_2, 2^{j+i})^{1/2}}{2^{\max(i,j)/2}} \left\|\Pi_{L_1}(u_i)\right\|_{L^2} \left\|\Pi_{L_2}(v_j)\right\|_{L^2} \quad \text{if} \quad |i - j| \geqslant 3 \end{aligned} \tag{11}$$

and

$$\left\|\Pi_{L_1}(u_i)\Pi_{L_2}(v_j)\right\|_{L^2} \lesssim L_1^{1/2} L_2^{1/4} \left\|\Pi_{L_1}(u_i)\right\|_{L^2} \left\|\Pi_{L_2}(v_j)\right\|_{L^2} \quad \text{if} \quad |i - j| < 3. \qquad \square \tag{12}$$

Let us show how (11) and (12) imply Lemma 10.
Observe first that

$$\left\|u_i v_j\right\|_{L^2} \leqslant \|u_i\|_{X^{0,1/2+\varepsilon}} \left\|v_j\right\|_{X^{0,1/2+\varepsilon}}. \tag{13}$$

Indeed, this follows by decomposing $u_i = \sum \Pi_{L_1}(u_i)\ v_j = \sum \Pi_{L_2}(v_j)$ and applying (11) and (12):

$$\begin{aligned}\left\|u_i v_j\right\|_{L^2} &\lesssim \sum_{L_1, L_2 \geqslant 1, dyadic} (L_1)^{1/2}(L_2)^{1/2} \left\|\Pi_{L_1}(u_i)\right\|_{L^2} \left\|\Pi_{L_2}(v_j)\right\|_{L^2} \\ &\lesssim \left(\sum_{L_1 \geqslant 1} (L_1)^{1+\varepsilon} \left\|\Pi_{L_1}(u_i)\right\|_{L^2}^2\right)^{1/2} \left(\sum_{L_2 \geqslant 1} (L_2)^{1+\varepsilon} \left\|\Pi_{L_2}(v_j)\right\|_{L^2}^2\right)^{1/2} \\ &\lesssim \|u_i\|_{X^{0,1/2+\varepsilon}} \left\|v_j\right\|_{X^{0,1/2+\varepsilon}}.\end{aligned}$$

Note also that since $\|u\bar{v}\|_{L^2} = \|uv\|_{L^2}$ we have that $\left\|u_i \bar{v_j}\right\|_{L^2} \leqslant \|u_i\|_{L^2} \left\|v_j\right\|_{L^2}$.

According to our previous remarks, the norm of $X^{0,b}$ can be realized by pairing with a function in the dual space $X^{0,-b}$. Thus, we are led to consider the four-linear forms

$$M_1(u, v, w, z) = \sum_{\max(i,j,k,l) - \min(i,j,k,l) > 5} \int u_i \bar{v}_j w_k \bar{z}_l \, dx\, dt$$

and

$$M_2(u, v, w, z) = \sum_{(i,j,k):\ \max(i,j,k,l) - \min(i,j,k,l) \leqslant 5} \int u_i \bar{v}_j w_k \bar{z}_l \, dx\, dt.$$

Consider M_1 first. Take additional decompositions, according to the operators Π_L. We have

$$\begin{aligned}M_1(u, v, w, z) = &\sum_{L_1, L_2, L_3, L_4 \geqslant 1} \sum_{\max(i,j,k,l) - \min(i,j,k,l) > 5} \\ &\times \int \Pi_{L_1}(u_i) \Pi_{L_2}(\bar{v}_j) \Pi_{L_3}(w_k) \Pi_{L_4}(\bar{z}_l) \, dx\, dt.\end{aligned}$$

By the condition $\max(i, j, k, l) - \min(i, j, k, l) > 5$, we conclude that for at least one of integers (i, j, k) (say i), we have $|i - l| \geqslant 3$. Applying Cauchy–Schwartz, (11) and (13), yields

$$\begin{aligned}
|M_1(u,v,w,z)| &\lesssim \sum_{L_1,L_4\geqslant 1} \left\|\Pi_{L_1}(u_i)\Pi_{L_4}(z_l)\right\|_{L^2} \left\|v_j\right\|_{X^{0,1/2+\varepsilon}} \|w_k\|_{X^{0,1/2+\varepsilon}} \\
&\lesssim \sum_{L_1,L_4\geqslant 1} \frac{{L_1}^{1/2}\min(L_4,2^{l+i})^{1/2}}{2^{\max(i,l)/2}} \left\|\Pi_{L_1}(u_i)\right\|_{L^2} \left\|\Pi_{L_4}(z_l)\right\|_{L^2} \\
&\quad \times \left\|v_j\right\|_{X^{0,1/2+\varepsilon}} \|w_k\|_{X^{0,1/2+\varepsilon}} \\
&\lesssim \sum_{L_4\geqslant 1} \frac{\min(L_4,2^{l+i})^{1/2}}{2^{\max(i,l)/2}} \left\|\Pi_{L_4}(z_l)\right\|_{L^2} \|u_i\|_{X^{0,1/2+\varepsilon}} \left\|v_j\right\|_{X^{0,1/2+\varepsilon}} \\
&\quad \times \|w_k\|_{X^{0,1/2+\varepsilon}}.
\end{aligned}$$

But, splitting the sum in $L_4 \geqslant 2^{l+i}$ and $L_4 < 2^{l+i}$ gives the estimate

$$\sum_{L_4\geqslant 2^{l+i}} \min(L_4,2^{l+i})^{1/2} \left\|\Pi_{L_4}(z_l)\right\|_{L^2} \lesssim 2^{6\varepsilon\max(i,l)} \|z_l\|_{X^{0,1/2-2\varepsilon}},$$

whereas

$$\begin{aligned}
\sum_{L_4\leqslant 2^{l+i}} \min(L_4,2^{l+i})^{1/2} \left\|\Pi_{L_4}(z_l)\right\|_{L^2} &\lesssim 2^{6\varepsilon\max(i,l)} \sum_{L_4\leqslant 2^{l+i}} L_4^{1/2-3\varepsilon} \left\|\Pi_{L_4}(z_l)\right\|_{L^2} \\
&\lesssim 2^{6\varepsilon\max(i,l)} \|z_l\|_{X^{0,1/2-2\varepsilon}}.
\end{aligned}$$

Put everything together to get

$$|M_1(u,v,w,z)| \lesssim 2^{-l(1/2-6\varepsilon)} \|u_i\|_{X^{0,1/2+\varepsilon}} \left\|v_j\right\|_{X^{0,1/2+\varepsilon}} \|w_k\|_{X^{0,1/2+\varepsilon}} \|z_l\|_{X^{0,1/2-2\varepsilon}},$$

since $l \leqslant \max(i,l)$.
Equivalently,

$$\left\|u_i\bar{v}_j w_k\right\|_{X^{0,-1/2+2\varepsilon}} \lesssim 2^{-l(1/2-6\varepsilon)} \|u_i\|_{X^{0,1/2+\varepsilon}} \left\|v_j\right\|_{X^{0,1/2+\varepsilon}} \|w_k\|_{X^{0,1/2+\varepsilon}}$$

as claimed.
For M_2, we use Cauchy–Schwartz and (12), to estimate

$$\begin{aligned}
|M_2(u,v,w,z)| &\lesssim \left\|u_{l-5\leqslant\cdot\leqslant l+5} v_{l-5\leqslant\cdot\leqslant l+5}\right\|_{L^2} \left\|w_{l-5\leqslant\cdot\leqslant l+5} z_{l-5\leqslant\cdot\leqslant l+5}\right\|_{L^2} \\
&\lesssim \sum_{L_1,\dots,L_4\geqslant 1} (L_1)^{1/2} \left\|\Pi_{L_1} u_{l-5\leqslant\cdot\leqslant l+5}\right\|_{L^2} (L_3)^{1/2} \left\|\Pi_{L_3} w_{l-5\leqslant\cdot\leqslant l+5}\right\|_{L^2} \\
&\quad \times (L_2)^{1/4} \left\|\Pi_{L_2} v_{l-5\leqslant\cdot\leqslant l+5}\right\|_{L^2} (L_4)^{1/4} \left\|\Pi_{L_4} z_{l-5\leqslant\cdot\leqslant l+5}\right\|_{L^2}.
\end{aligned}$$

It is easy to see that

$$\sum_{L_1 \geqslant 1} (L_1)^{1/2} \left\| \Pi_{L_1} u_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{L^2} \lesssim \left\| u_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{X^{0,1/2+\varepsilon}}$$

and similarly for v, w. Finally, since ε is sufficiently small, we have

$$\sum_{L_4 \geqslant 1} (L_4)^{1/4} \left\| \Pi_{L_4} z_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{L^2} \lesssim \left\| z_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{X^{0,1/2-2\varepsilon}}$$

Altogether,

$$\begin{aligned} |M_2(u, v, w, z)| \lesssim\ & \left\| u_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{X^{0,1/2+\varepsilon}} \left\| v_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{X^{0,1/2+\varepsilon}} \left\| w_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{X^{0,1/2+\varepsilon}} \\ & \times \left\| z_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{X^{0,1/2-2\varepsilon}}. \end{aligned}$$

In terms of the norms, we have

$$\begin{aligned} & \left\| \sum_{(i,j,k):\ \max(i,j,k,l)-\min(i,j,k,l) \leqslant 5} u_i v_j w_k \right\|_{X^{0,-1/2+2\varepsilon}} \\ & \lesssim \left\| u_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{X^{0,1/2+\varepsilon}} \left\| v_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{X^{0,1/2+\varepsilon}} \left\| w_{l-5 \leqslant \cdot \leqslant l+5} \right\|_{X^{0,1/2+\varepsilon}} \end{aligned}$$

as claimed.

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